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The Forces on a Solid Body in a Stream of Viscous Fluid

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II—The Forces on a Solid Body in a Stream of Viscous Fluid

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1—Introduction

The object of the present investigation is to obtain formulae for the lift and drag when a solid body of any shape is at rest in a stream of incompressible viscous fluid, but no limitation is imposed upon the magnitude of the stream velocity. It is convenient to start by giving a short account of previous work along the same lines.

The forces on a cylinder of any shape in a stream of viscous fluid have been discussed by Filon.* Writing U for the velocity of the stream, and u, v, w for the additional disturbance velocities, and neglecting terms of second order in the disturbance, Filon obtains a system of linear equations identical with those adopted by Oseen. These equations, however, are only assumed to be valid at a great distance from the cylinder, and not at the surface of the solid, as in the applications made by Oseen and other writers. The complete solution of the equations is obtained in the form of two series of typical solutions, in which the corresponding motion is respectively rotational and irrotational. The lift on the cylinder is found to be given by the same expression as in the Kutta-Joukowski theorem for a perfect fluid. Also the drag is found to be associated with a particular term in the solution, which corresponds to an inward flow along the tail and a compensating outward flow across a large contour.

The same equations and their solution were subsequently dealt with by Faxén.† Faxén, however, assumes that the equations are valid at the surface of the cylinder, and his results are therefore restricted to small values of the Reynolds number. Also he does not obtain expressions for the lift and drag in terms of physically significant quantities.

The corresponding problem in three dimensions has been treated in two papers by Goldstein,‡ who follows Filon in using the Oseen approximation at a great distance from the solid. In his first paper Goldstein discusses two series of solutions

* 'Proc. Roy. Soc.,' vol. 113, p. 7 (1926).
† 'Nova Acta R. Soc. Sci., Upsala' (1927).
‡ 'Proc. Roy. Soc.,' A, vol. 123, p. 216 (1929); vol. 131, p. 198 (1931).

of the equations. The first gives a series of particular integrals, in which irrotational velocities are associated with certain values of the pressure. In the second series, which is of the nature of a complementary function, the velocities are rotational, while the pressure does not appear. FILON's theorem, connecting the drag with the inflow in the tail, is shown to hold for the solutions discussed in this paper.

In the second paper, GOLDSTEIN investigates some more particular integrals. He shows that for certain values of the pressure, the expressions giving the corresponding irrotational velocities have singularities, and these singularities have to be cancelled by the addition of suitable rotational velocities. The theorem concerning the drag is found to be still true. Also, by a consideration of the orders of magnitude of the various terms in the velocities, a simple expression, in the form of an integral, is obtained for the lift.

Further progress with the evaluation of the lift cannot be made without a complete investigation of the solutions of the original equations. Such an investigation is also necessary to determine which solutions are associated with the drag and the inflow in the tail. These problems are discussed in the present paper.

GOLDSTEIN* has also treated separately the special case of motion with axial symmetry ; this case has also been discussed by DAHL,† whose results agree with GOLDSTEIN'S.

2—SUMMARY OF RESULTS

The discussion is based upon the system of linear equations used by GOLDSTEIN and, for the two-dimensional problem, by FILON. The complete solution of these equations is obtained in §§3–7 of the present paper. The solution is divided into a series of particular integrals, involving both the velocities and the pressure p, and a complementary function, involving rotational velocities only. The pressure satisfies LAPLACE'S equation, and in general it is possible to obtain a suitable particular integral by associating each assumed typical value of p with certain irrotational velocities. If, however, p is a sectorial harmonic, the expressions giving the corresponding velocities have singularities, and these singularities have to be cancelled by the addition of suitable rotational velocities. This part of the work is facilitated by the introduction of a certain associated Legendre function, which does not appear to have been studied before.

The complementary function is discussed completely for the first time, and is expressed in the form of four series of typical solutions.

It is found that, at a great distance from the body, the vorticity is insensible except in a certain region behind the body, which will be referred to as the wake.

The solutions of the equations are entirely independent except for the condition of no total flow across a large surface in the fluid. This condition is found to lead

* ' Proc. Roy. Soc.,' A, vol. 123, p. 225 (1929).
† ' Ark. Mat. Astr. Fys.,' vol. 21, No. 5 (1928).

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to a relation between the arbitrary constants associated with certain solutions possessing axial symmetry. One of the irrotational solutions corresponds to a uniform outflow across a large surface, and this outflow is compensated by an inflow in the tail arising from the rotational solutions which have axial symmetry.

The general solution of the equations of motion having been obtained, the contributions of the various terms to the lift and drag are then examined in detail. The drag is found to be associated with the irrotational solution which gives the uniform outflow mentioned above. The value obtained for the drag is in accordance with GOLDSTEIN's result, and with the theorem obtained by FILON for the two-dimensional case.

The axis of x being taken to coincide with the direction of the stream, it is found that the lifting forces in the directions of the y- and z-axes are each associated with one of the special solutions which occur when p is a sectorial harmonic; these solutions will be referred to as S_y and S_z respectively. Now it is well known that in practice the lifting force on a body is always associated with a system of trailing vortices in the fluid behind the body. We find that, for the solutions S_y and S_z, there are trailing vortices in the region at a great distance from the body for which the present theoretical treatment is valid. Although in this region the vorticity has become widely diffused, enough of its characteristics persist to give equal and opposite circulations round two complementary halves of the wake, which die out as $r^{-\frac{1}{2}}$ as we go away from the body, r denoting distance from an origin in the body. Further, for any solutions other than S_y and S_z, the corresponding circulations tend to zero more rapidly than $r^{-\frac{1}{2}}$. Thus the lifting forces are definitely connected with the circulations round the diffused trailing vortices at infinity, and it is also found that the relation between the signs of the lift and the circulations is that required by observation.

GOLDSTEIN* shows in his second paper that the solutions S_y and S_z give lifting forces, but it has not previously been shown that they are the only solutions which do so. Also, the present paper gives the first correct identification of the three-dimensional lift with a physically significant quantity.

The investigation is based upon the equations for steady motion, but the results can be extended to motion which is periodic in character, *i.e.*, steady on the average. This aspect of the matter is not discussed, however, as it has been fully dealt with by FILON,† for the two-dimensional case, and his treatment is equally applicable to the three-dimensional problem.

When the work on the complementary function was nearly finished, I discovered that the results of §4 had been obtained some years previously by Professor FILON, who, however, had not published them. Professor FILON kindly allowed me to check my work by comparison with his own, and the results were found to be in complete agreement.

* 'Proc. Roy. Soc.,' A, vol. 131, p. 198 (1931).

† 'Proc. Roy. Soc.,' A, vol. 113, p. 7 (1926).

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3—The Approximate Equations of Motion

We consider a stream of viscous fluid flowing past a fixed solid body of any shape, the velocity of the undisturbed stream being U parallel to the x-axis. Let $U+u$, v, w be the velocity resolutes at any point of the fluid, so that u, v, w are the divergences from the uniform stream. We assume that, at a great distance from the solid, terms of second order in the disturbance may be neglected. Then, if the motion is supposed steady, the hydrodynamical equations assume the form

$$\left.\begin{aligned} U\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x} &= \nu\nabla^2 u \\ U\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial y} &= \nu\nabla^2 v \\ U\frac{\partial w}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial z} &= \nu\nabla^2 w \end{aligned}\right\}, \quad \ldots \ldots \ldots \quad (3.1)$$

where p is the pressure, ρ the density, and ν the kinematic viscosity. We have also the equation of continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \quad \ldots \ldots \ldots \quad (3.2)$$

Equations of this form were first used by Oseen in his well-known solution for the moving sphere. They have since been applied by other writers to a number of problems in which they are supposed to hold at the surface of the solid, which means that U must be supposed small. In the present paper we shall only assume that the equations are valid at infinity, in which case there is no restriction on the value of U.

It follows from (3.1) and (3.2) that

$$\nabla^2 p = 0. \quad \ldots \ldots \ldots \quad (3.3)$$

We assume that both the pressure and the velocities of the disturbance tend to zero in all directions at a great distance from the solid, and we may therefore suppose p to be expanded in series of solid spherical harmonics of negative degree.

Consider now any typical value p_1 of the pressure. Suppose that u_1, v_1, w_1 are values of the velocities which, together with p_1, provide a solution of equations (3.1) and (3.2). Now let

$$u_1, + u_2,\ v_1 + v_2,\ w_1 + w_2$$

be any other values of the velocities which satisfy the same equations. Then u_2, v_2, w_2 must satisfy the equations

$$\left(\nabla^2 - 2k\frac{\partial}{\partial x}\right) u_2, v_2, w_2 = 0, \quad \ldots \ldots \ldots \quad (3.4)$$

where

$$k = \mathrm{U}/2\nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

together with (3.2).

The values u_1, v_1, w_1, and p_1 constitute a particular integral of equations (3.1) and (3.2), while u_1, v_2, w_2 form a complementary function, which is the same for all values of p. Thus the complete solution for the velocities is given by

$$u = u_1 + u_2, \; v = v_1 + v_2, \; w = w_1 + w_2,$$

provided that u_1, v_1, w_1 are now taken to mean the sum of the particular integrals corresponding to the various values of p.

In general, it is possible to obtain a suitable particular integral by associating a given typical value of p with certain irrotational velocities. If, however, p is a sectorial harmonic, the expressions giving the corresponding velocities have singularities, and these singularities have to be cancelled by the addition of suitable rotational velocities which satisfy (3.4). When dealing with the complementary function, however, we must confine ourselves to values of u_2, v_2, w_2 which tend to zero in all directions at a great distance from the solid, for there is here no possibility of cancelling singularities. The complementary function can therefore be expressed entirely in terms of well-known functions, but the special particular integrals just referred to are found to involve a new type of associated Legendre function. For this reason it is convenient to consider the complementary function first.

4—The Complementary Function

We proceed now to investigate the general solution of the system of equations (3.4) and (3.2). The solutions of the related systems of equations

$$\nabla^2 (u, v, w) = 0,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0;$$

and

$$(\nabla^2 + k^2)\,(u, v, w) = 0,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

are to be found in various works on mathematical physics, *e.g.*, LAMB's "Hydrodynamics". A new method will, however, be adopted for the solution of the system with which we are concerned.

It is convenient at this stage to introduce the following notation, r, θ, ω being spherical polar coordinates with the axis of x as polar axis.

$$C_n = r^{-(n+1)} P_n(\cos\theta),$$

$$C_n^m = r^{-(n+1)} P_n^m(\cos\theta)\cos m\omega,$$

$$D_n^m = r^{-(n+1)} P_n^m(\cos\theta)\sin m\omega.$$

$$F_n(kr) = \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} K_{n+\frac{1}{2}}(kr),$$

$$G_n = F_n(kr)\, P_n(\cos\theta),$$

$$G_n^m = F_n(kr)\, P_n^m(\cos\theta)\cos m\omega,$$

$$H_n^m = F_n(kr)\, P_n^m(\cos\theta)\sin m\omega.$$

$K_{n+\frac{1}{2}}$ is the Bessel function of imaginary argument, defined as in WATSON's "Bessel Functions" (§§3.7, 3.71). P_n^m is the associated Legendre function, defined by the equation

$$P_n^m(\mu) = (1-\mu^2)^{\frac{m}{2}} \frac{d^m P_n(\mu)}{d\mu^m},$$

μ denoting $\cos\theta$.

The functions C_n, C_n^m and D_n^m are solid spherical harmonics. They do not appear in the complementary function, which involves rotational velocities only, but they will be required later when we discuss the particular integrals.

The functions G_n, G_n^m and H_n^m are solutions of the equation

$$(\nabla^2 - k^2)\, V = 0, \quad \ldots \ldots \ldots \quad (4.01)$$

while the functions $e^{kx} G_n$, $e^{kx} G_n^m$ and $e^{kx} H_n^m$ satisfy the equation (3.4).

We are only concerned at present with those solutions of (3.4) which tend to zero as r tends to infinity, remain finite for all values of θ, and have period 2π in ω. The functions $e^{kx}G_n$, $e^{kx}G_n^m$, and $e^{kx}H_n^m$ provide all the solutions satisfying these conditions, if n takes all positive integral values, including zero, and m takes all positive integral values such that $m \leq n$, this restriction being due to the fact that $P_n^m(\cos\theta)$ vanishes if $m > n$.

The typical solutions for u_2, v_2, w_2 are the functions $e^{kx}G_n$, $e^{kx}G_n^m$, and $e^{kx}H_n^m$. Our problem is to find combinations of these solutions such that (3.2) is satisfied. With this object in view, we shall first obtain formulae expressing the partial derivatives of the typical solutions with respect to x, y, and z in terms of other typical solutions. With the help of these formulae we shall then obtain four series of values of u_2, v_2, w_2 which satisfy (3.2). Finally, we shall show that these four series include all possible solutions of the system of equations (3.4) and (3.2).

On differentiating the functions G_n, G_n^m, H_n^m with respect to x, y, and z, we meet with the following expressions involving Legendre functions:

$$\mu P_n(\mu), \qquad (1-\mu^2)\frac{dP_n(\mu)}{d\mu};$$

$$\mu P_n^m(\mu), \qquad (1-\mu^2)\frac{dP_n^m(\mu)}{d\mu};$$

$$(1-\mu^2)^{\frac{1}{2}} P_n(\mu), \qquad \mu(1-\mu^2)^{\frac{1}{2}}\frac{dP_n(\mu)}{d\mu};$$

$$(1-\mu^2)^{\frac{1}{2}} P_n^m(\mu), \qquad \mu(1-\mu^2)^{\frac{1}{2}}\frac{dP_n^m(\mu)}{d\mu}, \qquad \frac{P_n^m(\mu)}{(1-\mu^2)^{\frac{1}{2}}}.$$

The most convenient method of procedure is to express all these as linear combinations of Legendre functions.

For the first three expressions, we have the following well-known recurrence formulae:

$$(2n+1)\,\mu P_n = (n+1)\,P_{n+1} + nP_{n-1}, \quad \dots\dots\dots \quad (4.10)$$

$$(2n+1)(1-\mu^2)\frac{dP_n}{d\mu} = n(n+1)\,[P_{n-1} - P_{n+1}], \quad \dots\dots\dots \quad (4.11)$$

$$(2n+1)\,\mu P_n^m = (n+1-m)\,P_{n+1}^m + (n+m)\,P_{n-1}^m. \quad .\ . \quad (4.20)$$

To deal with $(1-\mu^2)\dfrac{dP_n^m(\mu)}{d\mu}$, we differentiate (4.11) m times, and multiply by $(1-\mu^2)^{\frac{1}{2}m}$, which gives

$$(2n+1)(1-\mu^2)\frac{dP_n^m}{d\mu} = (2n+1)\,m\mu P_n^m + n(n+1)\,[P_{n-1}^m - P_{n+1}^m]$$
$$+ (2n+1)\,m(m-1)(1-\mu^2)^{\frac{1}{2}} P_n^{m-1}. \quad .\ . \quad (4.21)$$

On differentiating $m-1$ times the formula

$$(2n+1)\,P_n = \frac{dP_{n+1}}{d\mu} - \frac{dP_{n-1}}{d\mu}, \quad \dots\dots\dots \quad (4.22)$$

and multiplying by $(1-\mu^2)^{\frac{1}{2}m}$, we get

$$(2n+1)(1-\mu^2)^{\frac{1}{2}} P_n^{m-1} = P_{n+1}^m - P_{n-1}^m. \quad \dots\dots \quad (4.23)$$

Making use of (4.20) and (4.23), (4.21) becomes

$$(2n+1)(1-\mu^2)\frac{dP_n^m}{d\mu} = (n+1)(n+m)\,P_{n-1}^m - n(n+1-m)\,P_{n+1}^m. \quad (4.24)$$

The expressions $(1-\mu^2)^{\frac{1}{2}} P_n(\mu)$ and $\mu(1-\mu^2)^{\frac{1}{2}} \frac{dP_n(\mu)}{d\mu}$ are easily dealt with. From (4.22) we have

$$(2n+1)(1-\mu^2)^{\frac{1}{2}} P_n = P_{n+1}{}^1 = P_{n-1}{}^1. \quad (4.30)$$

Also making use of (4.20), we have

$$(2n+1)\mu(1-\mu^2)^{\frac{1}{2}} \frac{dP_n}{d\mu} = nP_{n+1}{}^1 + (n+1) P_{n-1}{}^1. \quad (4.31)$$

Next, by changing m to $m+1$ in (4.23), we have

$$(2n+1)(1-\mu^2)^{\frac{1}{2}} P_n{}^m = P_{n+1}{}^{m+1} - P_{n-1}{}^{m+1}. \quad (4.40)$$

We now turn to the expression $\frac{P_n{}^m(\mu)}{(1-\mu^2)^{\frac{1}{2}}}$. By direct differentiation of $P_n{}^m(\mu)$, we have

$$\frac{dP_n{}^m}{d\mu} = \frac{1}{(1-\mu^2)^{\frac{1}{2}}} P_n{}^{m+1} - \frac{m\mu}{1-\mu^2} P_n{}^m. \quad (4.41)$$

From (4.41) and (4.24), we have, making use also of (4.20),

$$(2n+1)(1-\mu^2)^{\frac{1}{2}} P_n{}^{m+1} = (n+m)(n+1+m) P_{n-1}{}^m - (n-m)(n+1-m) P_{n+1}{}^m. \quad (4.42)$$

Eliminating $P_{n-1}{}^m$ from (4.42) and (4.23), and changing n to $n-1$, we get

$$2m \frac{P_n{}^m}{(1-\mu^2)^{\frac{1}{2}}} = P_{n-1}{}^{m+1} + (n+m)(n+m-1) P_{n-1}{}^{m-1}. \quad (4.43)$$

Also, eliminating $P_{n+1}{}^m$ and changing n to $n+1$, we have

$$2m \frac{P_n{}^m}{(1-\mu^2)^{\frac{1}{2}}} = P_{n+1}{}^{m+1} + (n+1-m)(n+2-m) P_{n+1}{}^{m-1}. \quad (4.44)$$

Finally, we have to consider the expression $\mu(1-\mu^2)^{\frac{1}{2}} \frac{dP_n{}^m(\mu)}{d\mu}$. From (4.24) we have

$$(2n+1)\mu(1-\mu^2)^{\frac{1}{2}} \frac{dP_n{}^m}{d\mu} = (n+1)(n+m) \frac{\mu P_{n-1}{}^m}{(1-\mu^2)^{\frac{1}{2}}} - n(n+1-m) \frac{\mu P_{n+1}{}^m}{(1-\mu^2)^{\frac{1}{2}}}.$$

Making use of (4.43) and (4.44), and afterwards of (4.20), we find that

$$2(2n+1)\mu(1-\mu^2)^{\frac{1}{2}} \frac{dP_n{}^m}{d\mu} = (n-m) P_{n+1}{}^{m+1} + (n+m+1) P_{n-1}{}^{m+1} - (n+m)(n+1-m)\{(n+2-m) P_{n+1}{}^{m-1} + (n+m-1) P_{n-1}{}^{m-1}.\} \quad (4.50)$$

There is a number of special cases of these formulae, in which certain terms disappear owing to the fact that n and m can only take positive values such that $m \leq n$.

We also meet with the expressions $\frac{F_n(kr)}{kr}$ and $F'_n(kr)$, which it is convenient to transform into linear combinations of the functions F_n. With the help of the well-known recurrence formulae

$$K_m(z) = \frac{z}{2m}[K_{m+1}(z) - K_{m-1}(z)],$$

and

$$K'_m(z) = -\tfrac{1}{2}[K_{m+1}(z) + K_{m-1}(z)],$$

we easily find that

$$(2n+1)\frac{F_n(kr)}{kr} = F_{n+1}(kr) - F_{n-1}(kr), \quad \ldots \quad (4.61)$$

and

$$(2n+1)F'_n(kr) = -[(n+1)F_{n+1}(kr) + nF_{n-1}(kr)]. \quad \ldots \quad (4.62)$$

The following special cases of these formulae may be noticed; they depend upon the fact that $K_{-\frac{1}{2}}(z) = K_{\frac{1}{2}}(z)$:

$$\frac{F_0(kr)}{kr} = F_1(kr) - F_0(kr), \quad \ldots \quad (4.63)$$

$$F'_0(kr) = -F_1(kr). \quad \ldots \quad (4.64)$$

We see that the formulae for F_0 are of the same form as those for F_n, except that when n is zero, F_{n-1} becomes F_0.

With the help of the above formulae for the F_n's and the P_n^m's, we readily obtain the following results:

$$\frac{\partial G_n}{\partial x} = \frac{-k}{2n+1}[nG_{n-1} + (n+1)G_{n+1}], \quad \ldots \quad (4.71)$$

$$\frac{\partial G_n^m}{\partial x} = \frac{-k}{2n+1}[(n+m)G_{n-1}^m + (n+1-m)G_{n+1}^m], \quad \ldots \quad (4.72)$$

$$\frac{\partial H_n^m}{\partial x} = \frac{-k}{2n+1}[(n+m)H_{n-1}^m + (n+1-m)H_{n+1}^m], \quad \ldots \quad (4.73)$$

$$\frac{\partial G_n}{\partial y} = \frac{k}{2n+1}[G_{n-1}^1 - G_{n+1}^1], \quad \ldots \quad (4.74)$$

$$\frac{\partial G_n^m}{\partial y} = \frac{k}{2(2n+1)}[G_{n-1}^{m+1} - G_{n+1}^{m+1} - (n+m)(n+m-1)G_{n-1}^{m-1} + (n+1-m)(n+2-m)G_{n+1}^{m-1}], \quad \ldots \quad (4.75)$$

$$\frac{\partial H_n^m}{\partial y} = \frac{k}{2(2n+1)}[H_{n-1}^{m+1} - H_{n+1}^{m+1} - (n+m)(n+m-1)H_{n-1}^{m-1} + (n+1-m)(n+2-m)H_{n+1}^{m-1}]. \quad \ldots \quad (4.76)$$

$$\frac{\partial G_n}{\partial z} = \frac{k}{2n+1} [H_{n-1}{}^1 - H_{n+1}{}^1], \quad (4.77)$$

$$\frac{\partial G_n{}^m}{\partial z} = \frac{k}{2(2n+1)} [H_{n-1}{}^{m+1} - H_{n+1}{}^{m+1} + (n+m)(n+m-1) H_{n-1}{}^{m-1} - (n+1-m)(n+2-m) H_{n+1}{}^{m-1}], \quad (4.78)$$

$$\frac{\partial H_n{}^m}{\partial z} = \frac{k}{2(2n+1)} [-G_{n-1}{}^{m+1} + G_{n+1}{}^{m+1} - (n+m)(n+m-1) G_{n-1}{}^{m-1} + (n+1-m)(n+2-m) G_{n+1}{}^{m-1}]. \quad (4.79)$$

A number of special cases of these formulae arises, which corresponds to the special cases of the formulae for the $P_n{}^m$'s mentioned above.

We now come to the problem of determining combinations of the typical solutions for u_2, v_2, w_2 which satisfy equation (3.2). It is easily seen from the differentiation formulae (4.71) to (4.79) that we cannot achieve this by combinations of single solutions. The results obtained by Filon* for the corresponding problem in two dimensions lead us to expect, however, that we can satisfy equation (3.2) with values of u_2, v_2, w_2 which each contain two typical solutions.

It is clear that the solutions will fall into two classes, in one of which u_2 and v_2 contain the functions $G_n{}^m$ and w_2 contains the functions $H_n{}^m$, while in the other class the position is reversed. We proceed to investigate the solutions of the first class.

We see from equations (4.71) to (4.79) that if we take as a trial solution

$$u_2 = \alpha_1 e^{kx} G_n{}^m + \alpha_2 e^{kx} G_{n+1}{}^m,$$
$$v_2 = \beta_1 e^{kx} G_n{}^{m+1} + \beta_2 e^{kx} G_{n+1}{}^{m+1},$$
$$w_2 = \gamma_1 e^{kx} H_n{}^{m+1} + \gamma_2 e^{kx} H_{n+1}{}^{m+1},$$

then div. (u_2, v_2, w_2) contains the eight functions

$$e^{kx} G_{n+r}{}^m, \qquad e^{kx} G_{n+r}{}^{m+2}, \qquad r = -1, 0, 1, 2.$$

The assumed values of u_2, v_2, w_2 will give a solution provided that we can choose the five ratios of the constants α_1, α_2, etc., so that the coefficients of these eight functions vanish. It is easily seen that we cannot choose values of u_2, v_2, w_2 such that div. $(u_2, v_2\ w_2)$ contains fewer than eight functions.

Now an inspection of equations (4.75) and (4.79) shows that the coefficients of $e^{kx} G_{n+r}{}^{m+2}$, $r = -1, 0, 1, 2$, will vanish if, and only if,

$$\gamma_1 = \beta_1, \qquad \gamma_2 = \beta_2.$$

Equating to zero the coefficients of $e^{kx} G_{n-1}{}^m$ and $e^{kx} G_{n+2}{}^m$, we then find that

$$\alpha_1 = -(n+m+1)\beta_1,$$
$$\alpha_2 = (n+1-m)\beta_2.$$

* 'Proc. Roy. Soc.,' A, vol. 113, p. 7 (1926).

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Finally, we find that the coefficients of $e^{kx}G_n^m$ and $e^{kx}G_{n+1}^m$ will both vanish provided that

$$\beta_1 + \beta_2 = 0.$$

Thus, if we write

$$-\beta_1 = \beta_2 = \alpha_n^m,$$

we obtain the solution

$$\left.\begin{aligned} u_2 &= \alpha_n^m e^{kx}\,[(n+m+1)\,G_n^m + (n+1-m)\,G_{n+1}^m], \\ v_2 &= \alpha_n^m e^{kx}\,[-G_n^{m+1} + G_{n+1}^{m+1}], \\ w_2 &= \alpha_n^m e^{kx}\,[-H_n^{m+1} + H_{n+1}^{m+1}]. \end{aligned}\right\} \quad . \; . \; (4.81)$$

By giving different values to n and m, we get a series of typical solutions for u_2, v_2, w_2 which satisfy equation (3.2) ; this series will be referred to as the solutions of type I.

In a similar manner we obtain also a series of solutions given by

$$\left.\begin{aligned} u_2 &= \beta_n^m e^{kx}\,[G_n^{m+1} + G_{n+1}^{m+1}], \\ v_2 &= \beta_n^m e^{kx}\,[(n+m+1)\,G_n^m - (n+1-m)\,G_{n+1}^m], \\ w_2 &= \beta_n^m e^{kx}\,[-(n+m+1)\,H_n^m + (n+1-m)\,H_{n+1}^m]. \end{aligned}\right\} \quad . \; . \; (4.82)$$

This series will be referred to as the solutions of type II.

As explained above, there is another class of solutions in which u_2 and v_2 contain the functions H_n^m and w_2 contains the functions G_n^m. In this case, also, the solutions fall into two series, which are given by

$$\left.\begin{aligned} u_2 &= \gamma_n^m e^{kx}\,[(n+m+1)\,H_n^m + (n+1-m)\,H_{n+1}^m], \\ v_2 &= \gamma_n^m e^{kx}\,[-H_n^{m+1} + H_{n+1}^{m+1}], \\ w_2 &= \gamma_n^m e^{kx}\,[G_n^{m+1} - G_{n+1}^{m+1}], \end{aligned}\right\} \quad . \; . \; (4.83)$$

and

$$\left.\begin{aligned} u_2 &= \delta_n^m e^{kx}\,[H_n^{m+1} + H_{n+1}^{m+1}], \\ v_2 &= \delta_n^m e^{kx}\,[(n+m+1)\,H_n^m - (n+1-m)\,H_{n+1}^m], \\ w_2 &= \delta_n^m e^{kx}\,[(n+m+1)\,G_n^m - (n+1-m)\,G_{n+1}^m]. \end{aligned}\right\} \quad . \; . \; (4.84)$$

These series will be referred to as the solution of types III and IV respectively.

There is a number of special cases of these solutions, which corresponds to the special cases of the differentiation formulae (4.71) to (4.79).

Some of these special cases are of considerable importance. Thus, if in the solutions of type I we put $m = 0$, we have

$$\begin{aligned} u_2 &= (n+1)\,\alpha_n e^{kx}\,[G_n + G_{n+1}], \\ v_2 &= \alpha_n e^{kx}\,[-G_n^1 + G_{n+1}^1], \\ w_2 &= \alpha_n e^{kx}\,[-H_n^1 + H_{n+1}^1]. \end{aligned}$$

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These are the only rotational solutions which are symmetrical about the axis of x. They have been used by OSEEN* and GOLDSTEIN† to discuss the flow of viscous liquid past a fixed sphere. We shall find in §9 that they are also the solutions which are associated with the drag in the general case.

If we put $m = 0$ in the solutions of type II, w_2 vanishes, and we have

$$u_2 = \beta_n e^{kx} [G_n^1 + G_{n+1}^1],$$

$$v_2 = (n+1) \beta_n e^{kx} [G_n - G_{n+1}],$$

$$w_2 = 0.$$

Similarly, by putting $m = 0$ in the solutions of types III and IV, we obtain solutions for which u_2 and v_2 vanish respectively. These solutions which involve two velocities only have been used by the author‡ to discuss the flow of viscous liquid past a rotating sphere.

It remains to show that the systems of solutions which we have obtained include all possible solutions. Considering those solutions in which u_2 and v_2 involve the functions G_n^m, and w_2 the functions H_n^m, we may suppose in the first place that u_2, v_2, w_2 are given by the following series of typical solutions :

$$u_2 = a_0 G_0 + \sum_{n=1}^{\infty} \{a_n G_n + \sum_{m=1}^{n} a_n^m G_n^m\},$$

$$v_2 = b_0 G_0 + \sum_{n=1}^{\infty} \{b_n G_n + \sum_{m=1}^{n} b_n^m G_n^m\},$$

$$w_2 = \sum_{n=1}^{\infty} \sum_{m=1}^{n} c_n^m H_n^m,$$

where, however, the coefficients a_n^m, b_n^m, c_n^m are not independent, since u_2, v_2, w_2 must satisfy (3.2). If we substitute these values of u_2, v_2, w_2, into the expression div. (u_2, v_2, w_2), and equate to zero the coefficients of the functions G_n^m, for $n \leq N$, we obtain $\frac{1}{2}(N+1)(N+2)$ relations between

$$(N+2)(N+3) + \tfrac{1}{2}(N+1)(N+2) - 1$$

coefficients. The latter number is made up of

$$\tfrac{1}{2}(N+2)(N+3) - 1 \ a_n^m\text{'s}, \ \tfrac{1}{2}(N+2)(N+3) \ b_n^m\text{'s},$$

and

$$\tfrac{1}{2}(N+1)(N+2) \ c_n^m\text{'s}.$$

Thus, if we choose arbitrarily $(N+2)(N+3) - 1$ of these coefficients, we can theoretically find the remaining $\frac{1}{2}(N+1)(N+2)$, though we may not be able

* "Hydrodynamik," Akademische Verlagsgesellschaft, Leipzig, 1927.

† 'Proc. Roy. Soc.,' A, vol. 123, p. 225 (1929).

‡ 'Proc. Roy. Soc.,' A, vol. 142, p. 491 (1933).

to express the result in a manageable form using the coefficients a_n^m, b_n^m, c_n^m. We therefore try to replace these coefficients by others so as to obtain a convenient form.

Suitable new coefficients are suggested by the series of solutions obtained above. In accordance with the equations (4.81) and (4.82), we write

$$a_n^m = (n-m)\,\alpha_{n-1}^m + (n+m+1)\,\alpha_n^m + \beta_{n-1}^{m-1} + \beta_n^{m-1},$$

$$b_n^m = \alpha_{n-1}^{m-1} - \alpha_n^{m-1} - (n-m)\,\beta_{n-1}^m + (n+m+1)\,\beta_n^m,$$

$$c_n^m = \alpha_{n-1}^{m-1} - \alpha_n^{m-1} + (n-m)\,\beta_{n-1}^m - (n+m+1)\,\beta_n^m,$$

with suitable modifications for $m = 0$ and $m = n$, equation (3.2) being then satisfied identically.

It is easily shown that the $(N+2)(N+3) + \frac{1}{2}(N+1)(N+2) - 1$ coefficients mentioned above have been replaced by $\frac{1}{2}(N+2)(N+3) - 1$ α_n^m's and $\frac{1}{2}(N+2)(N+3)$ β_n^m's, *i.e.*, $(N+2)(N+3) - 1$ coefficients in all. Thus our system of solutions contains the greatest possible number of independent coefficients, *i.e.*, the system obtained is the most general possible.

In the same way we can show that the two systems of solutions, in which u_2 and v_2 involve the functions H_n^m and w_2 the functions G_n^m, also include all possible solutions of this form.

5—The Associated Legendre Function $R_n^m(\mu)$

It has been seen in §3 that we may suppose the pressure p to be expanded in a series of solid spherical harmonics of negative degree. We shall find that when p is a sectorial harmonic, the corresponding values of v_1 and w_1 are most conveniently expressed in terms of a type of associated Legendre function which does not seem to have been studied before in the particular form which it is convenient to use here. It therefore seems best to give a short account of this function before discussing the particular integrals.

It is well known that the solution of Legendre's associated equation

$$(1-\mu^2)\,d^2v/d\mu^2 - 2\mu\,dv/d\mu + [n(n+1) - m^2(1-\mu^2)^{-1}]\,v = 0, \quad (5.1)$$

n and m being positive integers, and $|\mu| \leq 1$, is given by

$$v = AP_n^m(\mu) + BQ_n^m(\mu),$$

where

$$P_n^m(\mu) = (1-\mu^2)^{\frac{m}{2}}\,(d^m/d\mu^m)\,P_n(\mu), \quad \ldots \quad (5.11)$$

$$Q_n^m(\mu) = (1-\mu^2)^{\frac{m}{2}}\,(d^m/d\mu^m)\,Q_n(\mu) \quad \ldots \quad (5.12)$$

If $m > n$, $P_n^m(\mu)$ vanishes, and we are left with only one solution. The solution for $m > n$ has been given by HEINE* in the form

$$v = A(1-\mu^2)^{\frac{m}{2}}(d^{m-n-1}/d\mu^{m-n-1})\{(\mu^2-1)^{-(n+1)}\}$$
$$+ B(1-\mu^2)^{\frac{m}{2}}(d^{m-n-1}/d\mu^{m-n-1})\{(\mu^2-1)^{-(n+1)}\int_{\mu}^{1}(\mu^2-1)^n\,d\mu\}.$$

We shall now show how to obtain the solution for $m > n$ in a form which is more convenient for our present purpose, and which also fits in more naturally with the solution for $m \leq n$.

We start with the well-known formula

$$Q_n(\mu) = \tfrac{1}{2}P_n(\mu)\log(1+\mu/1-\mu) - W_{n-1},$$

where W_{n-1} is a polynomial in μ of degree $n-1$.

It follows from this that if $m > n-1$,

$$Q_n^m(\mu) = \tfrac{1}{2}(1-\mu^2)^{\frac{m}{2}}\{(d^m/d\mu^m)[P_n(\mu)\log(1+\mu)] - (d^m/d\mu^m)[P_n(\mu)\log(1-\mu)]\}.$$

We shall now show that if $m > n$, the expressions

$$(1-\mu^2)^{\frac{m}{2}}(d^m/d\mu^m)[P_n(\mu)\log(1\pm\mu)]$$

each satisfy (5.1), and may thus be taken as the two independent solutions. If we write

$$R_n(\mu) = P_n(\mu)\log(1-\mu),$$

and use the fact that $P_n(\mu)$ is a solution of LEGENDRE's equation

$$(1-\mu^2)\,d^2v/d\mu^2 - 2\mu\,dv/d\mu + n(n+1)\,v = 0,$$

we find that $R_n(\mu)$ satisfies the equation

$$(1-\mu^2)\,d^2v/d\mu^2 - 2\mu\,dv/d\mu + n(n+1)\,v = -2(1+\mu)\,dP_n/d\mu - P_n.$$

The right-hand side is a polynomial of degree n in μ, so that differentiating m times, we see that if $m > n$, $(d^m/d\mu^m)R_n(\mu)$ satisfies the equation

$$(1-\mu^2)\,d^2v/d\mu^2 - 2(m+1)\,\mu\,dw/d\mu + (n-m)(n+m+1)\,w = 0. \quad (5.2)$$

Now it is known that if w is any solution of (5.2), then $(1-\mu^2)^{\frac{m}{2}}w$ satisfies (5·1). Thus $(1-\mu^2)^{\frac{m}{2}}(d^m/d\mu^m)\{P_n(\mu)\log(1-\mu)\}$ is a solution of (5.1) if $m > n$, and similarly it can be shown that $(1-\mu^2)^{\frac{m}{2}}(d^m/d\mu^m)\{P_n(\mu)\log(1+\mu)\}$

* 'Kugelfunctionen,' p. 153.

is also a solution. It may be noticed that these functions contain no logarithmic term, since $P_n(\mu)$ is a polynomial of degree n.

Also, whereas the function $Q_n^m(\mu)$ becomes infinite when $\mu = \pm 1$, the new solutions each become infinite for only one of these values of μ. In discussing the particular integrals we shall only require the solution

$$(1-\mu^2)^{\frac{m}{2}} (d^m/d\mu^m) \{P_n(\mu) \log (1-\mu)\},$$

which will be denoted by $R_n^m(\mu)$. This function is defined for all positive integral values of m and n, but it is only of interest when $m > n$, since it is only then that it constitutes a solution of equation (5·1).

Since $P_n(\mu)$ is a polynomial in μ of degree n, it is clear that, if $m > n$, $(d^m/d\mu^m)\{P_n(\mu) \log (1-\mu)\}$ can be expressed as a polynomial in $(1-\mu)^{-1}$. The form of this polynomial may be obtained as follows. If in (5.1) we make the substitution

$$(1-\mu)^{-1} = \lambda,$$

we obtain the equation

$$\lambda^2 (2\lambda - 1)\, d^2v/d\lambda^2 + 2[(1-m)\lambda^2 + m\lambda]\, dv/d\lambda + (n-m)(n+m+1)\, v = 0. \tag{5.3}$$

Assuming a solution in series of the form

$$v = \sum_{r=p}^{\infty} a_r \lambda^r,$$

we find the following relation between successive coefficients:

$$(r+n-m)(r-n-m-1)\, a_r = 2(r-1)(r-m-1)\, a_{r-1}.$$

The indicial equation is

$$(p+n-m)(p-n-m-1) = 0.$$

If we take $p = m-n$, then provided $m > n$, we obtain a solution of equation (5.3) in the form

$$v = A\left\{\lambda^{m-n} + \frac{2(m-n)\,n}{1!\,2n}\lambda^{m-n+1} + \frac{2^2(m-n)(m-n+1)\,n(n-1)}{2!\,2n(2n-1)}\lambda^{m-n+2} + \dots\right\}$$

$$= A\lambda^{m-n} F(m-n, -n, -2n, 2\lambda),$$

in the usual hypergeometric notation. The series terminates when we get to the term in λ^m, and is thus a polynomial in λ containing $n+1$ terms. This is the required solution, and we see that if $m > n$, then

$$(d^m/d\mu^m)\{P_n(\mu) \log (1-\mu)\} = A(1-\mu)^{-(m-n)} F[m-n, -n, -2n, 2/(1-\mu)]. \tag{5.4}$$

The value of A may be determined by finding the coefficient of $(1-\mu)^{-(m-n)}$ in $(d^m/d\mu^m)\{P_n(\mu)\log(1-\mu)\}$. In order to do this, we shall first find the coefficient of $(1-\mu)^{-1}$ in $(d^{n+1}/d\mu^{n+1})\{P_n(\mu)\log(1-\mu)\}$. We have

$$\begin{aligned}(d^{n+1}/d\mu^{n+1})&\{P_n(\mu)\log(1-\mu)\}\\ &= -n!\,(1-\mu)^{-(n+1)}P_n(\mu)-(n+1)(n-1)!\,(1-\mu)^{-n}(d/d\mu)P_n(\mu)\\ &\quad-(n+1)\,n\,(n-2)!\,(1-\mu)^{-(n-1)}(d^2/d\mu^2)P_n(\mu)/2!\,\ldots\\ &\quad-(n+1)\,n\ldots 2\,(1-\mu)^{-1}(d^n/d\mu^n)P_n(\mu)/n!.\end{aligned}$$

The coefficient of $(1-\mu)^{-1}$ in this expression is equal to the sum of the coefficients of the highest powers of μ in the successive numerators, taken with alternate signs. If n is even, the first term is negative, while if n is odd the first term is positive. With the help of RODRIGUE's formula

$$P_n(\mu)=(d^n/d\mu^n)(\mu^2-1)^n/2^n\,n!,$$

we find that the coefficient of $(1-\mu)^{-1}$ is

$$(-1)^{n+1}\frac{2n(2n-1)\ldots(n+1)}{2^n}\left\{1-(n+1)+\frac{(n+1)\,n}{2!}\ldots+(-1)^n\frac{(n+1)\,n\ldots 2}{n!}\right\}$$
$$=-2n(2n-1)\ldots(n+1)/2^n.$$

It follows that the coefficient of $(1-\mu)^{-(m-n)}$ in

$(d^m/d\mu^m)\{P_n(\mu)\log(1-\mu)\}$ is

$$-(m-n-1)!\,2n(2n-1)\ldots(n+1)/2^n,$$

and this gives the value of A.

Finally, from (5.4) we have

$$R_n^m(\mu)=A(1-\mu^2)^{\frac{m}{2}}(1-\mu)^{-(m-n)}F[m-n,-n,-2n,2/(1-\mu)]. \quad . \quad (5.5)$$

It may be shown that, if m and n are positive integers such that $m>n$, the expression on the right-hand side of (5.5) is equivalent, apart from a numerical factor, to a generalized definition of $P_n^m(\mu)$, given by HOBSON.* It is not, however, convenient to use HOBSON's formula, and we shall continue to use the symbol $R_n^m(\mu)$, in order to distinguish the functions which become infinite when $\mu=1$ from the functions $P_n^m(\mu)$, which remain finite for all values of μ such that $|\mu|\leq 1$.

* "Spherical and Ellipsoidal Harmonics," p. 227.

The formula (5.5) is convenient for calculating actual values of $R_n^m(\mu)$. A list of some of the early values is given below :

$$R_0^1(\mu) = -(1-\mu^2)^{\frac{1}{2}}(1-\mu)^{-1},$$

$$R_0^2(\mu) = -(1-\mu^2)(1-\mu)^{-2},$$

$$R_0^3(\mu) = -2(1-\mu^2)^{\frac{3}{2}}(1-\mu)^{-3},$$

$$R_1^2(\mu) = -(1-\mu^2)[(1-\mu)^{-1} + (1-\mu)^{-2}],$$

$$R_1^3(\mu) = -(1-\mu^2)^{\frac{3}{2}}[(1-\mu)^{-2} + 2(1-\mu)^{-3}],$$

$$R_2^3(\mu) = -(1-\mu^2)^{\frac{3}{2}}[3(1-\mu)^{-1} + 3(1-\mu)^{-2} + 2(1-\mu)^{-3}],$$

$$R_3^4(\mu) = -(1-\mu^2)^2[15(1-\mu)^{-1} + 15(1-\mu)^{-2} + 12(1-\mu)^{-3} + 6(1-\mu)^{-4}].$$

6—Solutions of Differential Equations Containing the Factor $R_n^m(\cos\theta)$

We now introduce the following notation, which is a natural extension of that used in §4.

$$\mathbf{C}_n^m = r^{-(n+1)} R_n^m(\cos\theta)\cos m\omega,$$

$$\mathbf{D}_n^m = r^{-(n+1)} R_n^m(\cos\theta)\sin m\omega,$$

$$\mathbf{G}_n^m = F_n(kr) R_n^m(\cos\theta)\cos m\omega,$$

$$\mathbf{H}_n^m = F_n(kr) R_n^m(\cos\theta)\sin m\omega.$$

These functions all become infinite when $\theta = 0$, since they contain the factor $R_n^m(\cos\theta)$, and bold type is used to distinguish them from the functions of §4, which contain the factor $P_n(\cos\theta)$ or $P_n^m(\cos\theta)$, and which therefore remain finite for all values of θ.

The new functions are of no importance if $m \leq n$, but if $m > n$, $\mathbf{C}_n^m$ and $\mathbf{D}_n^m$ are solid spherical harmonics, while $\mathbf{G}_n^m$ and $\mathbf{H}_n^m$ are solutions of equation (4.01), and $e^{kx}\mathbf{G}_n^m$ and $e^{kx}\mathbf{H}_n^m$ satisfy equation (3.4).

In order to discuss the particular integrals, it is necessary to obtain formulae for the partial derivatives of the new functions with respect to x, y and z.

On differentiating the functions $\mathbf{C}_n^m$, etc., we meet with the same series of expressions involving the functions $\mathbf{R}_n^m(\mu)$ as those involving $\mathbf{P}_n^m(\mu)$ which were met with in §4. As before, it is convenient to express these as linear combinations of Legendre functions. We find that if $m > n+1$ (in some cases if $m > n+2$), the $\mathbf{R}_n^m$'s satisfy the same recurrence formulae as the $\mathbf{P}_n^m$'s. We shall, however, require a number of special cases in which m is equal to n or $n+1$, and it is found here that both $\mathbf{P}_n^m$'s and $\mathbf{R}_n^m$'s appear in the recurrence formulae. It will not be necessary to discuss these cases in detail, since they only involve the application of the formulae for the P_n^m's obtained in §4.

Consider first the expression $\mu R_n^m(\mu)$. Multiplying (4.10) by log $(1-\mu)$ and differentiating m times, we have

$$(2n+1)\,\mu\,\frac{d^mR_n}{d\mu^m}+(2n+1)\,m\,\frac{d^{m-1}R_n}{d\mu^{m-1}}=(n+1)\,\frac{d^mR_{n+1}}{d\mu^m}+n\,\frac{d^mR_{n-1}}{d\mu^m}. \quad (6.01)$$

From (4.22) and (4.11) we get

$$(2n+1)\,\mu R_n=\frac{dR_{n+1}}{d\mu}-\frac{dR_{u-1}}{d\mu}-\frac{(2n+1)}{n\,(n+1)}\,(1+\mu)\,\frac{dP_n}{d\mu}. \quad . \quad . \quad (6.02)$$

The last term on the right-hand side of (6.02) is a polynomial of degree n, and so if we differentiate $m-1$ times, where $m>n+1$, we have

$$(2n+1)\,\frac{d^{m-1}R_n}{d\mu^{m-1}}=\frac{d^mR^{n+1}}{d\mu^m}-\frac{d^mR_{n-1}}{d\mu^m} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (6.03)$$

Eliminating $\frac{d^{m-1}R_n}{d\mu^{m-1}}$ between (6.01) and (6.03), we get

$$(2n+1)\,\mu R_n^m=(n+1-m)\,R_{n+1}^m+(n+m)\,R_{n-1}^m. \quad . \quad . \quad . \quad (6.04)$$

If $m\leq n+1$, this result is modified. We require the following cases:

$$(2n+1)\,\mu R_n^{n+1}=(2n+1)\,R_{n-1}^{n+1}+P_{n+1}^{n+1}, \quad . \ . \ . \ . \ . \quad (6.05)$$

$$(2n+1)\,\mu R_n^n=R_{n+1}^n+2nR_{n+1}^n+\frac{2n+1}{n+1}\,P_n^n+\frac{n}{n+1}\,P_{n+1}^n. \quad . \quad . \quad (6.06)$$

It will be noticed that this discussion fails altogether if n is zero, since equation (6.02) does not apply. A separate treatment of this case will be given later.

To deal with $(1-\mu^2)\,\frac{dR_n^m(\mu)}{d\mu}$, we notice that (4.11) leads to the equation

$$(2n+1)\,(1-\mu^2)\,\frac{dR_n}{d\mu}+(2n+1)\,(1+\mu)P_n=n(n+1)\,(R_{n-1}-R_{n+1}).$$

On differentiating this m times, the term $(1+\mu)\,P_n$ disappears if $m>n+1$. The discussion then proceeds as for $(1-\mu^2)^{\frac{1}{2}}\,\frac{dP_n^m(\mu)}{d\mu}$, making use of (6.03) instead of (4.23). We find that if $m>n+1$,

$$(2n+1)\,(1-\mu^2)\,\frac{dR_n^m}{d\mu}=(n+1)\,(n+m)R_{n-1}^m-n(n+1-m)R_{n+1}^m. \quad . \quad (6.07)$$

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As before, the result is modified if $m \leq n+1$; we require the following cases:

$$(2n+1)(1-\mu^2)\frac{dR_n^{n+1}}{d\mu} = (n+1)(2n+1)R_{n-1}^{n+1} - nP_{n+1}^{n+1}, \quad . \quad (6.08)$$

$$(2n+1)(1-\mu^2)\frac{dR_n^{n}}{d\mu} = 2n(n+1)R_{n-1}^{n} - nR_{n+1}^{n} - \frac{n(2n+1)}{n+1}P_n^n - \frac{(n^2+n+1)}{n+1}P_{n+1}^n \quad . \quad (6.09)$$

Next, by changing m to $m+1$ in (6.03), we have

$$(2n+1)(1-\mu^2)^{\frac{1}{2}}R_n^m = R_{n+1}^{m+1} - R_{n-1}^{m+1}. \quad . \; . \; . \; . \; . \quad (6.10)$$

This formula holds provided $m > n$. We also have

$$(2n+1)(1-\mu^2)^{\frac{1}{2}}R_n^n = R_{n+1}^{n+1} - R_{n-1}^{n+1} - \frac{1}{n+1}P_{n+1}^{n+1}, \quad . \; . \; . \quad (6.11)$$

$$(2n+1)(1-\mu^2)^{\frac{1}{2}}R_n^{n-1} = R_{n+1}^{n} - R_{n-1}^{n} - \frac{(2n+1)}{n(n+1)}P_n^n - \frac{1}{n+1}P_{n+1}^n. \quad . \quad (6.12)$$

We now turn to the expression $\dfrac{R_n^m(\mu)}{(1-\mu^2)^{\frac{1}{2}}}$. Proceeding exactly as for $\dfrac{P_n^m(\mu)}{(1-\mu^2)^{\frac{1}{2}}}$, we find that, if $m > n$,

$$\frac{2m R_n^m}{(1-\mu^2)^{\frac{1}{2}}} = R_{n-1}^{m+1} + (n-1+m)(n+m)R_{n-1}^{m-1}, \quad . \; . \; . \; . \quad (6.13)$$

and, if $m > n+2$,

$$\frac{2m R_n^m}{(1-\mu^2)^{\frac{1}{2}}} = R_{n+1}^{m+1} + (n+1-m)(n+2-m)R_{n+1}^{m-1}. \quad . \; . \quad (6.14)$$

We also have

$$\frac{2(n+2)R_n^{n+2}}{(1-\mu^2)^{\frac{1}{2}}} = R_{n+1}^{n+3} - P_{n+1}^{n+1}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6.15)$$

$$\frac{2(n+1)R_n^{n+1}}{(1-\mu^2)^{\frac{1}{2}}} = R_{n+1}^{n+2} + P_{n+1}^{n}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6.16)$$

Finally, we have to consider the expression $\mu(1-\mu^2)^{\frac{1}{2}}\dfrac{dR_n^m(\mu)}{d\mu}$. As in the case of $\mu(1-\mu^2)^{\frac{1}{2}}\dfrac{dP_n^m(\mu)}{d\mu}$, we have, if $m > n+2$,

$$2(2n+1)\mu(1-\mu^2)^{\frac{1}{2}}\frac{dR_n^m}{d\mu} = (n-m)R_{n+1}^{m+1} + (n+1+m)R_{n-1}^{m+1} - (n+m)(n+1-m)\{(n+2-m)R_{n+1}^{m-1} + (n+m-1)R_{n-1}^{m-1}\}. \quad . \quad (6.17)$$

We also find that

$$2\,(2n+1)\,\mu(1-\mu^2)^{\frac{1}{2}}\frac{dR_n^{\,n+2}}{d\mu} = -\,2\,R_{n+1}^{\,n+3} + (2n+3)\,R_{n-1}^{\,n+3} + 2(n+1)\,\{(2n+1)R_{n-1}^{\,n+1} + P_{n+1}^{\,n+1}\}, \quad (6.18)$$

$$2\,(2n+1)\mu\,(1-\mu^2)^{\frac{1}{2}}\frac{dR_n^{\,n+1}}{d\mu} = -\,R_{n+1}^{\,n+2} + 2(n+1)\,R_{n-1}^{\,n+2} - (2n+1)P_{n+1}^{\,n}. \quad (6.19)$$

We now have to investigate the form taken by these equations when n is equal to zero. The most convenient procedure is to make use of equation (5.5). Putting n equal to zero, this becomes

$$R_0^{\,m}(\mu) = \frac{-\,(m-1)\,!}{(1-\mu)^m}\,(1-\mu^2)^{m/2}. \quad (6.20)$$

With the help of (6.20) we readily obtain the following results :

$$\mu R_0^{\,m} = mR_0^{\,m} - (m-1)\,R_1^{\,m}, \qquad (m>1) \quad (6.21)$$

$$\mu R_0^{\,1} = R_0^{\,1} - P_1^{\,1}, \quad (6.22)$$

$$(1-\mu^2)\,\frac{dR_0^{\,m}}{d\mu} = mR_0^{\,m}, \quad (6.23)$$

$$(1-\mu^2)^{\frac{1}{2}}\,R_0^{\,m} = R_1^{\,m+1} - R_0^{\,m+1}. \quad (6.30)$$

Equations (6.23) and (6.30) hold for all values of m.

$$\frac{2mR_0^{\,m}}{(1-\mu^2)^{\frac{1}{2}}} = R_0^{\,m+1} + (m-1)\,mR_0^{\,m-1}, \qquad (m>1) \quad (6.31)$$

$$\frac{2mR_0^{\,m}}{(1-\mu^2)^{\frac{1}{2}}} = R_1^{\,m+1} + (m-1)\,(m-2)\,R_1^{\,m-1}, \qquad (m>2) \quad (6.32)$$

$$\frac{4R_0^{\,2}}{(1-\mu^2)^{\frac{1}{2}}} = R_1^{\,3} - P_1^{\,1}, \quad (6.33)$$

$$\frac{2R_0^{\,1}}{(1-\mu^2)^{\frac{1}{2}}} = R_0^{\,2} - P_0 \quad (6.34)$$

$$= R_1^{\,2} + P_1. \quad (6.35)$$

$$2\mu\,(1-\mu^2)^{\frac{1}{2}}\frac{dR_0^{\,m}}{d\mu} = (m+1)\,R_0^{\,m+1} - mR_1^{\,m+1} + m\,(m-1)\,\{(m-1)\,R_0^{\,m-1} - (m-2)\,R_1^{\,m-1}\}, \qquad (m>2) \quad (6.36)$$

$$2\mu\,(1-\mu^2)^{\frac{1}{2}}\frac{dR_0^{\,2}}{d\mu} = 3R_0^{\,3} - 2R_1^{\,3} + 2\,(R_0^{\,1} + P_1^{\,1}), \quad (6.37)$$

$$2\mu\,(1-\mu^2)^{\frac{1}{2}}\frac{dR_0^{\,1}}{d\mu} = 2R_0^{\,2} - R_1^{\,2} - P_1. \quad (6.38)$$

It will be noticed that the formulae for the R_0^{m}'s are of the same form as those for the R_n^{m}'s, except that when n is zero, R_{n-1}^m becomes R_0^m. The same peculiarity was found to occur in the recurrence formulae for the F_n's obtained in §4.

With the help of these formulae for the R_n^{m}'s, and also the formulae for the P_n^{m}'s and the F_n's obtained in §4, we obtain the following results, which are required for the discussion of the particular integrals :

$$\frac{\partial C_{n-1}}{\partial x} = -nC_n, \qquad (6.41)$$

$$\frac{\partial C_{n-1}^m}{\partial x} = -(n-m)\, C_n^m, \qquad (6.42)$$

$$\frac{\partial \mathbf{C}_{n-1}^n}{\partial x} = -C_n^n, \qquad (6.43)$$

$$\frac{\partial D_{n-1}^m}{\partial x} = -(n-m)\, D_n^m, \qquad (6.44)$$

$$\frac{\partial \mathbf{D}_{n-1}^n}{\partial x} = -D_n^n, \qquad (6.45)$$

$$\left.\begin{aligned} \frac{\partial}{\partial x} e^{kx}\, \mathbf{G}_0^1 &= -ke^{kx}\, G_1^1, \\ \frac{\partial}{\partial x} e^{kx}\, \mathbf{G}_0^n &= -k(n-1)\, e^{kx} [\mathbf{G}_0^n - \mathbf{G}_1^n], \qquad (n>1) \end{aligned}\right\} \qquad (6.46)$$

$$\left.\begin{aligned} \frac{\partial}{\partial x} e^{kx}\, \mathbf{G}_r^n &= -\frac{ke^{kx}}{2r+1} [(r+n)\, \mathbf{G}_{r-1}^n - (2r+1)\, \mathbf{G}_r^n \\ &\qquad + (r+1-n)\mathbf{G}_{r+1}^n], \qquad (0<r<n-1) \\ \frac{\partial}{\partial x} e^{kx}\, \mathbf{G}_{n-1}^n &= -\frac{ke^{kx}}{2n-1} [(2n-1)\, (\mathbf{G}_{n-2}^n - \mathbf{G}_{n-1}^n) + G_n^n], \quad (n>1). \end{aligned}\right\} \qquad (6.47)$$

$$\left.\begin{aligned} \frac{\partial}{\partial x} e^{kx}\, \mathbf{H}_0^1 &= -ke^{kx}\, H_1^1, \\ \frac{\partial}{\partial x} e^{kx}\, \mathbf{H}_0^n &= -k(n-1)\, e^{kx} [\mathbf{H}_0^n - \mathbf{H}_1^n], \qquad (n>1) \end{aligned}\right\} \qquad (6.48)$$

$$\left.\begin{aligned} \frac{\partial}{\partial x} e^{kx}\, \mathbf{H}_r^n &= -\frac{ke^{kx}}{2r+1} [(r+n)\, \mathbf{H}_{r-1}^n - (2r+1)\, \mathbf{H}_r^n \\ &\qquad + (r+1-n)\, \mathbf{H}_{r+1}^n], \qquad (0<r<n-1) \\ \frac{\partial}{\partial x} e^{kx}\, \mathbf{H}_{n-1}^n &= -\frac{ke^{kx}}{2n-1} [(2n-1)\, (\mathbf{H}_{n-2}^n - \mathbf{H}_{n-1}^n) + H_n^n], \quad (n>1). \end{aligned}\right\} \qquad (6.49)$$

$$\left.\begin{aligned}\frac{\partial C_n}{\partial y} &= -C_{n+1}{}^1,\\ \frac{\partial C_n{}^m}{\partial y} &= \tfrac{1}{2}[(n+1-m)(n+2-m)C_{n+1}{}^{m-1}-C_{n+1}{}^{m+1}],\\ \frac{\partial D_n{}^m}{\partial y} &= \tfrac{1}{2}[(n+1-m)(n+2-m)D_{n+1}{}^{m-1}-D_{n+1}{}^{m+1}].\end{aligned}\right\} \quad (6.51)$$

$$\left.\begin{aligned}\frac{\partial C_n}{\partial z} &= -D_{n+1}{}^1,\\ \frac{\partial C_n{}^m}{\partial z} &= -\tfrac{1}{2}[(n+1-m)(n+2-m)D_{n+1}{}^{m-1}+D_{n+1}{}^{m+1}],\\ \frac{\partial D_n{}^m}{\partial z} &= \tfrac{1}{2}[(n+1-m)(n+2-m)C_{n+1}{}^{m-1}+C_{n+1}{}^{m+1}].\end{aligned}\right\} \quad (6.52)$$

$$\frac{\partial \mathbf{C}_0{}^1}{\partial y} = \tfrac{1}{2}[C_1-\mathbf{C}_1{}^2], \quad (6.61)$$

$$\frac{\partial \mathbf{C}_{n-1}{}^n}{\partial y} = \tfrac{1}{2}[C_n{}^{n-1}-\mathbf{C}_n{}^{n+1}], \qquad (n>1) \quad (6.62)$$

$$\frac{\partial \mathbf{G}_0{}^1}{\partial y} = \frac{k}{2}[\mathbf{G}_0{}^2-\mathbf{G}_1{}^2-G_0+G_1], \quad (6.63)$$

$$\frac{\partial \mathbf{G}_0{}^2}{\partial y} = \frac{k}{2}[\mathbf{G}_0{}^3-\mathbf{G}_1{}^3-2\mathbf{G}_0{}^1-G_1{}^1], \quad (6.64)$$

$$\frac{\partial \mathbf{G}_0{}^n}{\partial y} = \frac{k}{2}[\mathbf{G}_0{}^{n+1}-\mathbf{G}_1{}^{n+1}-n(n-1)\mathbf{G}_0{}^{n-1} + (n-1)(n-2)\mathbf{G}_1{}^{n-1}], \qquad (n>2) \quad (6.65)$$

$$\frac{\partial \mathbf{G}_r{}^n}{\partial y} = \frac{k}{2(2r+1)}[\mathbf{G}_{r-1}{}^{n+1}-\mathbf{G}_{r+1}{}^{n+1}-(r+n)(r+n-1)\mathbf{G}_{r-1}{}^{n-1} + (n-r-1)(n-r-2)\mathbf{G}_{r+1}{}^{n-1}], \quad (6.66)$$

$$\frac{\partial \mathbf{G}_{n-2}{}^n}{\partial y} = \frac{k}{2(2n-3)}[\mathbf{G}_{n-3}{}^{n+1}-\mathbf{G}_{n-1}{}^{n+1}-(2n-2)(2n-3)\mathbf{G}_{n-3}{}^{n-1} - G_{n-1}{}^{n-1}], \qquad (n>2) \quad (6.67)$$

$$\frac{\partial \mathbf{G}_{n-1}{}^n}{\partial y} = \frac{k}{2(2n-1)}[\mathbf{G}_{n-2}{}^{n+1}-\mathbf{G}_n{}^{n+1}-(2n-1)(2n-2)\mathbf{G}_{n-2}{}^{n-1} + G_n{}^{n-1}], \qquad (n>1). \quad (6.68)$$

$$\frac{\partial \mathbf{D}_0{}^1}{\partial y} = -\tfrac{1}{2}\mathbf{D}_1{}^2, \quad (6.71)$$

$$\frac{\partial \mathbf{D}_{n-1}{}^n}{\partial y} = \tfrac{1}{2}[D_n{}^{n-1}-\mathbf{D}_n{}^{n+1}], \qquad (n>1) \quad (6.72)$$

$$\frac{\partial \mathbf{H}_0^1}{\partial y} = \frac{k}{2}[\mathbf{H}_0^2 - \mathbf{H}_1^2], \qquad (6.73)$$

$$\frac{\partial \mathbf{H}_0^2}{\partial y} = \frac{k}{2}[\mathbf{H}_0^3 - \mathbf{H}_1^3 - 2\mathbf{H}_0^1 - \mathrm{H}_1^1], \qquad (6.74)$$

$$\frac{\partial \mathbf{H}_0^n}{\partial y} = \frac{k}{2}[\mathbf{H}_0^{n+1} - \mathbf{H}_1^{n+1} - n(n-1)\,\mathbf{H}_0^{n-1} + (n-1)(n-2)\,\mathbf{H}_1^{n-1}], \quad (n > 2) \qquad (6.75)$$

$$\frac{\partial \mathbf{H}_r^n}{\partial y} = \frac{k}{2(2r+1)}[\mathbf{H}_{r-1}^{n+1} - \mathbf{H}_{r+1}^{n+1} - (r+n)(r+n-1)\,\mathbf{H}_{r-1}^{n-1} + (n-r-1)(n-r-2)\,\mathbf{H}_{r+1}^{n-1}], \qquad (6.76)$$

$$\frac{\partial \mathbf{H}_{n-2}^n}{\partial y} = \frac{k}{2(2n-3)}[\mathbf{H}_{n-3}^{n+1} - \mathbf{H}_{n-1}^{n+1} - (2n-2)(2n-3)\,\mathbf{H}_{n-3}^{n-1} - \mathrm{H}_{n-1}^{n-1}], \quad (n > 2) \qquad (6.77)$$

$$\frac{\partial \mathbf{H}_{n-1}^n}{\partial y} = \frac{k}{2(2n-1)}[\mathbf{H}_{n-2}^{n+1} - \mathbf{H}_n^{n+1} - (2n-1)(2n-2)\,\mathbf{H}_{n-2}^{n-1} + \mathrm{H}_n^{n-1}], \quad (n > 1). \qquad (6.78)$$

$$\frac{\partial \mathbf{C}_0^1}{\partial z} = -\tfrac{1}{2}\,\mathbf{D}_1^2, \qquad (6.81)$$

$$\frac{\partial \mathbf{C}_{n-1}^n}{\partial z} = -\tfrac{1}{2}[\mathrm{D}_n^{n-1} + \mathbf{D}_n^{n+1}], \quad (n > 1), \qquad (6.82)$$

$$\frac{\partial \mathbf{G}_0^1}{\partial z} = \frac{k}{2}[\mathbf{H}_0^2 - \mathbf{H}_1^2], \qquad (6.83)$$

$$\frac{\partial \mathbf{G}_0^2}{\partial z} = \frac{k}{2}[\mathbf{H}_0^3 - \mathbf{H}_1^3 + 2\mathbf{H}_0^1 + \mathrm{H}_1^1], \qquad (6.84)$$

$$\frac{\partial \mathbf{G}_0^n}{\partial z} = \frac{k}{2}[\mathbf{H}_0^{n+1} - \mathbf{H}_1^{n+1} + n(n-1)\,\mathbf{H}_0^{n-1} - (n-1)(n-2)\,\mathbf{H}_1^{n-1}], \quad (n > 2), \qquad (6.85)$$

$$\frac{\partial \mathbf{G}_r^n}{\partial z} = \frac{k}{2(2r+1)}[\mathbf{H}_{r-1}^{n+1} - \mathbf{H}_{r+1}^{n+1} + (r+n)(r+n-1)\,\mathbf{H}_{r-1}^{n-1} - (n-r-1)(n-r-2)\,\mathbf{H}_{r+1}^{n-1}], \qquad (6.86)$$

$$\frac{\partial \mathbf{G}_{n-2}^n}{\partial z} = \frac{k}{2(2n-3)}[\mathbf{H}_{n-3}^{n+1} - \mathbf{H}_{n-1}^{n+1} + (2n-3)(2n-2)\,\mathbf{H}_{n-3}^{n-1} + \mathrm{H}_{n-1}^{n-1}], \quad (n > 2), \qquad (6.87)$$

$$\frac{\partial \mathbf{G}_{n-1}^n}{\partial z} = \frac{k}{2(2n-1)}[\mathbf{H}_{n-2}^{n+1} - \mathbf{H}_n^{n+1} + (2n-2)(2n-1)\,\mathbf{H}_{n-2}^{n-1} - \mathrm{H}_n^{n-1}], \quad (n > 1). \qquad (6.88)$$

$$\frac{\partial \mathbf{D}_0^1}{\partial z} = \tfrac{1}{2}[\mathrm{C}_1 + \mathbf{C}_1^2], \qquad (6.91)$$

$$\frac{\partial \mathbf{D}_{n-1}{}^{n}}{\partial z} = \tfrac{1}{2}\left[\mathrm{C}_{n}^{n-1} + \mathbf{C}_{n}{}^{n+1}\right], \qquad (n > 1), \quad \ldots \quad (6.92)$$

$$\frac{\partial \mathbf{H}_{0}{}^{1}}{\partial z} = \frac{k}{2}\left[-\mathbf{G}_{0}{}^{2} + \mathbf{G}_{1}{}^{2} + \mathrm{G}_{0} + \mathrm{G}_{1}\right], \quad \ldots \quad (6.93)$$

$$\frac{\partial \mathbf{H}_{0}{}^{2}}{\partial z} = \frac{k}{2}\left[-\mathbf{G}_{0}{}^{3} + \mathbf{G}_{1}{}^{3} - 2\mathbf{G}_{0}{}^{1} - \mathrm{G}_{1}{}^{1}\right], \quad \ldots \quad (6.94)$$

$$\frac{\partial \mathbf{H}_{0}{}^{n}}{\partial z} = \frac{k}{2}\left[-\mathbf{G}_{0}{}^{n+1} + \mathbf{G}_{1}{}^{n+1} - n(n-1)\,\mathbf{G}_{0}{}^{n-1} + (n-1)(n-2)\,\mathbf{G}_{1}{}^{n-1}\right], \qquad (n > 2), \quad (6.95)$$

$$\frac{\partial \mathbf{H}_{r}{}^{n}}{\partial z} = \frac{k}{2(2r+1)}\left[-\mathbf{G}_{r-1}{}^{n+1} + \mathbf{G}_{r+1}{}^{n+1} - (r+n)(r+n-1)\,\mathbf{G}_{r-1}{}^{n-1} - (n-r-1)(n-r-2)\,\mathbf{G}_{r+1}{}^{n-1}\right], \quad (6.96)$$

$$\frac{\partial \mathbf{H}_{n-2}{}^{n}}{\partial z} = \frac{k}{2(2n-3)}\left[-\mathbf{G}_{n-3}{}^{n+1} + \mathbf{G}_{n-1}{}^{n+1} - (2n-3)(2n-2)\,\mathbf{G}_{n-3}{}^{n-1} - \mathrm{G}_{n-1}{}^{n-1}\right], \qquad (n > 2), \quad (6.97)$$

$$\frac{\partial \mathbf{H}_{n-1}{}^{n}}{\partial z} = \frac{k}{2(2n-1)}\left[-\mathbf{G}_{n-2}{}^{n+1} + \mathbf{G}_{n}{}^{n+1} - (2n-2)(2n-1)\,\mathbf{G}_{n-2}{}^{n-1} + \mathrm{G}_{n}^{n-1}\right], \qquad (n > 1). \quad (6.98)$$

Equations (6.66), (6.76), (6.86), and (6.96) hold for all values of r and n such that $0 < r < n-2$.

7—The Particular Integrals

We are now in a position to investigate the particular integrals of the system of equations (3.1) and (3.2). The method adopted combines certain features of both the methods described by Goldstein; the introduction of the function $\mathrm{R}_n^m(\mu)$, however, enables us to give a more systematic account of the matter than is otherwise possible.

We remark firstly that a particular integral of equations (3.1) is given by

$$u_1, v_1, w_1 = -\operatorname{grad}\phi, \quad \ldots \quad (7.11)$$

$$p = \rho \mathrm{U}\,\frac{\partial \phi}{\partial x}, \quad \ldots \quad (7.12)$$

and (3.2) and (3.3) are satisfied provided

$$\nabla^2 \phi = 0. \quad \ldots \quad (7.13)$$

The typical values for p are the functions C_n, C_n^m, D_n^m. If we start with any typical value of p, a suitable particular integral is given by equations (7.11) and (7.12), provided that the corresponding values of ϕ and its derivatives are finite for all values of θ and ω.

The next step is to determine the values of ϕ corresponding to the various possible values of p. As explained in GOLDSTEIN's paper,* the value

$$p = -\rho U/r,$$

cannot occur, since the corresponding value

$$\phi = \log (r - x)$$

leads to an infinite flow across a large surface in the fluid. To deal with the other values of p, we make use of equations (6.41) to (6.45), which lead to the series of corresponding values of ϕ and p given in Table I.

TABLE I

$-p/\rho U$	ϕ		
nC_n	C_{n-1}	$(n > 0)$	(7.21)
$(n - m)\, C_n^m$	C_{n-1}^m	$(m < n)$	(7.22)
C_n^n	$\mathbf{C}_{n-1}^n$		(7.23)
$(n - m)\, D_n^m$	D_{n-1}^m	$(m < n)$	(7.24)
D_n^n	$\mathbf{D}_{n-1}^n$		(7.25)

We see that if p is a zonal or tesseral harmonic, *i.e.*, $m < n$, ϕ and its derivatives are finite for all values of θ and ω, and it follows that suitable expressions for u_1 v_1, w_1, are given by (7.11). The actual values of u_1, v_1, w_1 in these cases can be readily written down with the help of equations (6.41), (6.42), (6.44), (6.51), and (6.52). These solutions of equations (3.1) and (3.2) will be referred to as the irrotational solutions.

When p is a sectorial harmonic, *i.e.*, $m = n$, the corresponding values of ϕ, $\partial\phi/\partial y$, and $\partial\phi/\partial z$ become infinite when $\theta = 0$. In these cases we shall therefore express the velocities giving the particular integral in the form

$$u_1 = -\frac{\partial\phi}{\partial x} + u_3,$$

$$v_1 = -\frac{\partial\phi}{\partial y} + v_3 + v_4,$$

$$w_1 = -\frac{\partial\phi}{\partial z} + w_3 + w_4,$$

where u_3, v_3, w_3, v_4, and w_4 all satisfy (3.4). Further, u_3, etc., will be chosen so that the expressions

$$-\frac{\partial\phi}{\partial x} + u_3, \qquad -\frac{\partial\phi}{\partial y} + v_3, \qquad -\frac{\partial\phi}{\partial z} + w_3,\ v_4, \text{ and } w_4,$$

* 'Proc. Roy. Soc.,' A, vol. 131, p. 198 (1931).

all remain finite when $\theta = 0$, while

$$\frac{\partial u_3}{\partial x} + \frac{\partial (v_3 + v_4)}{\partial y} + \frac{\partial (w_3 + w_4)}{\partial z} = 0. \qquad (7.30)$$

In this way a particular integral satisfying all the necessary conditions is obtained. These solutions of equations (3.1) and (3.2) will be referred to as the special solutions.

We shall now show how to determine the values of u_3, etc., corresponding to the values of p and ϕ given by (7.23). First, we proceed to find a function χ_n satisfying (4.01) and such that $\mathbf{C}_{n-1}{}^n + e^{kx}\chi_n$ remains finite for $\theta = 0$. Now $\mathbf{C}_{n-1}{}^n$ is of the form $f(r, \theta) \cos n\omega$, and, therefore, apart from the possible addition of a function which remains finite for $\theta = 0$, $e^{kx}\chi_n$ must be of the form $g(r, \theta) \cos n\omega$, *i.e.*, χ_n must be of the form $\sum_{r=0}^{n-1} \mathrm{A}_r{}^n \mathbf{G}_r{}^n$.

χ_1 is easily determined, for we have

$$\mathbf{C}_0{}^1 = \frac{-y}{r(r-x)}, \qquad (7.31)$$

$$e^{kx}\mathbf{G}_0{}^1 = -\frac{e^{-k(r-x)}}{kr}\frac{y}{r-x}. \qquad (7.32)$$

Thus the function $\mathbf{C}_0{}^1 - ke^{kx}\mathbf{G}_0{}^1$ remains finite for all values of θ, *i.e.*, $\chi_1 = -k\mathbf{G}_0{}^1$. The other functions χ_n may be determined as follows.

By hypothesis

$$\mathbf{C}_{n-1}{}^n + e^{kx}\chi_n \text{ is finite for } \theta = 0.$$

Hence also

$$\frac{\partial}{\partial y}(\mathbf{C}_{n-1}{}^n + e^{kx}\chi_n) \text{ is finite for } \theta = 0.$$

But from (6.61) and (6.62) we have

$$\frac{\partial}{\partial y}\mathbf{C}_{n-1}{}^n = -\tfrac{1}{2}\mathbf{C}_n{}^{n+1} + \text{an expression which is finite for } \theta = 0.$$

It follows that

$$\frac{\partial \chi_n}{\partial y} = -\tfrac{1}{2}\chi_{n+1} + \text{an expression which is finite for } \theta = 0.$$

$$= -\tfrac{1}{2}\sum_{r=0}^{n} \mathrm{A}_r{}^{n+1}\mathbf{G}_r{}^{n+1} + \text{an expression which is finite for } \theta = 0. \qquad (7.33)$$

With the help of equations (6.65) to (6.68), $\partial\chi_n/\partial y$ can be expressed in terms of the functions $\mathbf{G}_r{}^{n+1}$, $r = 0, 1, 2, \dots n$, and $\mathbf{G}_r{}^{n-1}$, $r = 0, 1, 2, \dots n-2$, which all become infinite when $\theta = 0$, and the functions $\mathrm{G}_{n-1}{}^{n-1}$ and $\mathrm{G}_n{}^{n-1}$, which remain finite for $\theta = 0$.

Substituting this value of $\partial\chi_n/\partial y$ into (7·33), and equating coefficients of $\mathbf{G}_r^{n+1}$, we obtain, if $n > 2$, the following equations connecting the coefficients A_r^n:

$$A_o^{n+1} = -k\left(A_o^n + \frac{A_1^n}{3}\right) \quad \ldots \quad (7.41)$$

$$A_r^{n+1} = -k\left(\frac{A_{r+1}^n}{2r+3} - \frac{A_{r-1}^n}{2r-1}\right), \ (1 \leq r \leq n-2) \quad \ldots \quad (7.42)$$

$$A_{n-1}^{n+1} = \frac{kA_{n-2}^n}{2n-3}; \qquad A_n^{n+1} = \frac{k\,A_{n-1}^n}{2n-1} \quad \ldots \quad (7.43)$$

Also, it follows from (7.33) that $\partial\chi_n/\partial y$ can contain no terms with the factor $\cos(n-1)\,\omega$ which become infinite when $\theta = 0$, and, therefore, the coefficients of $\mathbf{G}_r^{n-1}$ must vanish. This gives us the additional equations

$$\frac{(1+n)}{3} A_1^n = (1-n)\,A_o^n \quad \ldots \quad (7.44)$$

$$\frac{(r+1+n)\,(r+n)}{2r+3} A_{r+1}^n = \frac{(r+1-n)\,(r-n)}{2r-1} A_{r-1}^n, \qquad (1 \leq r \leq n-2). \quad (7.45)$$

Using (7.44) and (7.45), we find that

$$A_r^n = (-1)^r\,(2r+1)\,\frac{n!\,(n-1)!}{(n+r)!\,(n-r-1)!}\,A_o^n.$$

Also, from (7.41) and (7.44) we get

$$A_o^{n+1} = \frac{-2k}{n+1} A_o^n, \qquad (n > 2),$$

and using equations (6.63) and (6.64), we find that this formula is true for $n = 1$ and $n = 2$, whence

$$A_o^n = (-1)^{n-1}\,\frac{(2k)^{n-1}}{n!}\,A_o^1.$$

We have already shown that $A_o^1 = -k$, so that

$$A_o^n = (-1)^n\,\frac{2^{n-1}\,k^n}{n!},$$

and

$$A_r^n = (-1)^{r+n} 2^{n-1}\,k^n\,\frac{(2r+1)\,(n-1)!}{(n+r)!\,(n-r-1)!} \quad \ldots \quad (7.51)$$

The work may be checked by verifying that the expression found for A_r^n satisfies (7.42) and (7.43).

Finally we have

$$\chi_n = \sum_{r=0}^{n-1} (-1)^{r+n}\,2^{n-1}k^n\,\frac{(2r+1)\,(n-1)!}{(n+r)!\,(n-r-1)!}\,\mathbf{G}_r^n. \quad \ldots \quad (7.52)$$

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Now it is clear that if ψ_n satisfies (4.01) and is of the form

$$\chi_n + \text{an expression which is finite for } \theta = 0,$$

the expression $\mathbf{C}_{n-n}{}^n + e^{kx}\,\psi_n$ and its partial derivatives with respect to x, y, and z remain finite for $\theta = 0$. We shall take

$$\psi_n = -2\frac{\partial \chi_{n-1}}{\partial y} \quad \ldots \ldots \ldots \ldots \quad (7.53)$$

$$= \chi_n + \text{an expression which is finite for } \theta = 0,$$

and further, we now write

$$u_3, v_3, w_3 = -\operatorname{grad} e^{kx}\psi_n. \quad \ldots \ldots \ldots \quad (7.54)$$

We choose these values of u_3, v_3, w_3, rather than

$$u_3, v_3, w_3 = -\operatorname{grad} e^{kx}\chi_n,$$

for convenience in determining the expressions v_4, w_4 which have to be associated with u_3, v_3, w_3 in order to obtain possible fluid velocities.

Now since $\dfrac{\partial \mathbf{C}_{n-1}{}^n}{\partial x}$ and $\dfrac{\partial}{\partial x}(\mathbf{C}_{n-1}{}^n + e^{kx}\chi_n)$ remain finite for $\theta = 0$, it follows that $\dfrac{\partial}{\partial x}(e^{kx}\chi_n)$ is finite for $\theta = 0$. It is easily seen that suitable values of v_4 and w_4 are given by

$$v_4 = -4k\frac{\partial}{\partial x}(e^{kx}\chi_{n-1}),\ w_4 = 0. \quad \ldots \ldots \ldots \quad (7.55)$$

For v_4 is finite for $\theta = 0$, and further

$$\frac{\partial u_3}{\partial x} + \frac{\partial}{\partial y}(v_3 + v_4) + \frac{\partial}{\partial z}(w_3 + w_4) = -2k\frac{\partial}{\partial x}(e^{kx}\psi_n) - 4k\frac{\partial^2}{\partial x\,\partial y}(e^{kx}\chi_{n-1})$$
$$= 0$$

so that (7.30) is satisfied.

With the help of equations (6.46) and (6.47) we find that

$$v_4 = -\frac{(n-2)!}{(2n-3)!}2^n k^{n+1} e^{kx} \mathrm{G}_{n-1}{}^{n-1}. \quad \ldots \ldots \ldots \quad (7.56)$$

It follows that when $p = -\rho \mathrm{U}\mathrm{C}_n{}^n$, a suitable particular integral of equations (3.1) and (3.2) is given by

$$u_1 = -\frac{\partial \mathbf{C}_{n-1}{}^n}{\partial x} - \frac{\partial}{\partial x}(e^{kx}\psi_n), \quad \ldots \ldots \ldots \ldots \quad (7.61)$$

$$v_1 = -\frac{\partial \mathbf{C}_{n-1}{}^n}{\partial y} - \frac{\partial}{\partial y}(e^{kx}\psi_n) - \frac{(n-2)!}{(2n-3)!}2^n k^{n+1} e^{kx} \mathrm{G}_{n-1}{}^{n-1}, \quad \ldots \quad (7.62)$$

$$w_1 = -\frac{\partial \mathbf{C}_{n-1}{}^n}{\partial z} - \frac{\partial}{\partial z}(e^{kx}\psi_n). \quad \ldots \ldots \ldots \ldots \quad (7.63)$$

Making use of (7.53) and (7.43), we find that

$$\psi_n = \sum_{r=0}^{n-1} A_r{}^n \mathbf{G}_r{}^{\boldsymbol{n}} + A_{n-2}{}^n G_{n-2}{}^{n-2} - A_{n-1}{}^n G_{n-1}{}^{n-2}, \quad \ldots \quad (7.64)$$

the value of $A_r{}^n$ being given by (7·51). The actual values of u_1, v_1, w_1 can be written down with the help of equations (6.43), (6.46), (6.47), (6.62) to (6.68), and (6.82) to (6.88).

The case in which p and ϕ are given by (7.25) may be dealt with in the same way. The equations (6.48), (6.49), and (6.71) to (6.78) correspond exactly to (6.46), (6.47), and (6.61) to (6.68), except that there are no terms in (6.71) and (6.73) which remain finite for $\theta = 0$. It follows that if

$$\psi'_n = \sum_{r=0}^{n-1} A_r{}^n \mathbf{H}_r{}^{\boldsymbol{n}} + A_{n-2}{}^n H_{n-2}{}^{n-2} - A_{n-1}{}^n H_{n-1}{}^{n-2}, \quad \ldots \quad (7.65)$$

the expression $\mathbf{D}_{n-1}{}^{n} + e^{kx}\psi'_n$ and its partial derivatives with respect to x, y, and z remain finite for all values of θ. Writing

$$u_3, v_3, w_3 = -\operatorname{grad} e^{kx}\psi'_n, \quad \ldots \quad (7.66)$$

and

$$v_4 = -\frac{(n-2)!}{(2n-3)!} 2^n k^{n+1} e^{kx} H_{n-1}{}^{n-1}, \; w_4 = 0,$$

we obtain a suitable particular integral of equations (3.1) and (3.2) in the form

$$u_1 = -\frac{\partial \mathbf{D}_{n-1}{}^{n}}{\partial x} - \frac{\partial}{\partial x}(e^{kx}\psi'_n), \quad \ldots \quad (7.67)$$

$$v_1 = -\frac{\partial \mathbf{D}_{n-1}{}^{n}}{\partial y} - \frac{\partial}{\partial y}(e^{kx}\psi'_n) - \frac{(n-2)!}{(2n-3)!} 2^n k^{n+1} e^{kx} H_{n-1}{}^{n-1}, \quad \ldots \quad (7.68)$$

$$w_1 = -\frac{\partial \mathbf{D}_{n-1}{}^{n}}{\partial z} - \frac{\partial}{\partial z}(e^{kx}\psi'_n). \quad \ldots \quad (7.69)$$

The above discussion fails if $n = 1$, since the function χ_{n-1} is only defined for values of $n > 1$. This case is of considerable importance, as it is the solution associated with the lift, and we shall therefore discuss it in detail. We have to find the velocities corresponding to the values

$$p = -\rho U C_1{}^1, \; \phi = \mathbf{C}_0{}^1.$$

Now we already know that the expression $\mathbf{C}_0{}^1 - ke^{kx}\mathbf{G}_0{}^1$ remains finite for $\theta = 0$, and we therefore write, in this case

$$u_3, v_3, w_3 = \operatorname{grad} ke^{kx}\mathbf{G}_0{}^1.$$

Also we have

$$ke^{kx}\,\mathbf{G}_0{}^1 = \frac{\partial \psi}{\partial y},$$

where

$$\psi = \int_{r-x}^{\infty} \frac{e^{-k\lambda}}{\lambda}\, d\lambda.$$

It follows that suitable values of v_4 and w_4 are given by

$$v_4 = -2k\frac{\partial \psi}{\partial x} = -2k^2 e^{kx}\mathrm{G}_0, \qquad w_4 = 0.$$

For v_4 is finite for $\theta = 0$, and further

$$\frac{\partial u_3}{\partial x} + \frac{\partial}{\partial y}(v_3 + v_4) + \frac{\partial}{\partial z}(w_3 + w_4) = 2k\frac{\partial^2 \psi}{\partial x \partial y} - 2k\frac{\partial^2 \psi}{\partial y \partial x} = 0.$$

The complete solution, when written out in full with the help of equations (6.43), (6.46), (6.61), (6.63), (6.81), and (6.83), is found to be

$$u_1 = -\frac{\partial}{\partial x}(\mathbf{C}_0{}^1 - ke^{kx}\,\mathbf{G}_0{}^1) \quad \ldots \quad (7.71)$$

$$= \mathrm{C}_1{}^1 - k^2 e^{kx}\,\mathrm{G}_1{}^1, \quad \ldots \quad (7.72)$$

$$v_1 = -\frac{\partial}{\partial y}(\mathbf{C}_0{}^1 - ke^{kx}\,\mathbf{G}_0{}^1) - 2k^2 e^{kx}\,\mathrm{G}_0 \quad \ldots \quad (7.73)$$

$$= \tfrac{1}{2}[\mathrm{C}_1 - \mathbf{C}_1{}^2] + \frac{k^2}{2}e^{kx}[\mathbf{G}_0{}^2 - \mathbf{G}_1{}^2 - 3\mathrm{G}_0 + \mathrm{G}_1], \quad \ldots \quad (7.74)$$

$$w_1 = -\frac{\partial}{\partial z}(\mathbf{C}_0{}^1 - ke^{kx}\mathbf{G}_0{}^1) \quad \ldots \quad (7.75)$$

$$= \tfrac{1}{2}\mathbf{D}_1{}^2 + \frac{k^2}{2}e^{kx}[\mathbf{H}_0{}^2 - \mathbf{H}_1{}^2]. \quad \ldots \quad (7.76)$$

The case in which p and ϕ are given by

$$p = -\rho \mathrm{U}\mathrm{D}_1{}^1, \qquad \phi = \mathbf{D}_0{}^1.$$

may be dealt with in a similar manner. We now write

$$u_3, v_3, w_3 = \operatorname{grad} ke^{kx}\,\mathbf{H}_0{}^1,$$

$$v_4 = 0,\ w_4 = -2k^2 e^{kx}\mathrm{G}_0.$$

The complete solution in this case is given by

$$u_1 = -\frac{\partial}{\partial x}(\mathbf{D}_0^1 - ke^{kx}\mathbf{H}_0^1) \quad \ldots \quad (7.81)$$

$$= D_1^1 - k^2 e^{kx} H_1^1, \quad \ldots \quad (7.82)$$

$$v_1 = -\frac{\partial}{\partial y}(\mathbf{D}_0^1 - ke^{kx}\mathbf{H}_0^1) \quad \ldots \quad (7.83)$$

$$= \tfrac{1}{2}\mathbf{D}_1^2 + \frac{k^2}{2}e^{kx}[\mathbf{H}_0^2 - \mathbf{H}_1^2], \quad \ldots \quad (7.84)$$

$$w_1 = -\frac{\partial}{\partial z}(\mathbf{D}_0^1 - ke^{kx}\mathbf{H}_0^1) - 2k^2 e^{kx} G_0 \quad \ldots \quad (7.85)$$

$$= -\tfrac{1}{2}[C_1 + \mathbf{C}_1^2] + \frac{k^2}{2}e^{kx}[-\mathbf{G}_0^2 + \mathbf{G}_1^2 - 3G_0 + G_1] \quad \ldots \quad (7.86)$$

The solutions given by equations (7.71) to (7.76) and (7.81) to (7.86) have been discussed, in terms of different notations, by Oseen* and Goldstein.†

It is to be noticed that since the functions $e^{kx}\psi$, $e^{kx}\psi'$, $e^{kx}\mathbf{G}_0^1$ and $e^{kx}\mathbf{H}_0^1$ are all single-valued, the expressions u_3, v_3, w_3 are in all cases the gradient of a single-valued function. This fact is of importance with reference to the calculation of the lift in section 10.

8—The Condition for No Total Flow

The solutions of equations (3.1) and (3.2) obtained in §§ 4 and 7 are entirely independent, except for one condition, namely, that there can be no total flow across any large surface in the fluid. Consider now a large sphere Σ, which is everywhere at a great distance from the solid S, and whose radius r is ultimately made infinite. Then the condition for no total flow is

$$\lim_{r\to\infty}\iint_\Sigma \{l'(U+u) + m'v + n'w\}\,d\Sigma = 0. \quad \ldots \quad (8.11)$$

where l', m', n' are direction cosines of the outward-drawn normal to Σ. Since $\iint_\Sigma l'U\,d\Sigma$ vanishes identically, (8.11) may be replaced by

$$\lim_{r\to\infty}\iint_\Sigma (l'u + m'v + n'w)\,d\Sigma = 0. \quad \ldots \quad (8.12)$$

This condition implies a relation between some of the arbitrary constants associated with the various solutions of equations (3.1) and (3.2). In order to obtain this

* "Hydrodynamik," Akademische Verlagsgesellschaft, Leipzig, 1927, pp. 31–33.

† 'Proc. Roy. Soc.,' A, vol. 131, p. 198 (1931).

relation, it is necessary to find the contributions of the different solutions to the integral

$$\iint_{\Sigma} (l'u + m'v + n'w)\, d\Sigma = \mathrm{F}, \text{say.} \quad \ldots \ldots \ldots \quad (8.13)$$

Now all the terms in u, v, and w, which satisfy equation (3.4), contain the factor $e^{-k(r-x)}$ or $e^{-2kr\sin^2\frac{1}{2}\theta}$, and are thus exponentially small when r is large, except in the region for which θ is of order less than $(kr)^{-\frac{1}{2}}$. This region, which is on the downstream side of the body, will be referred to as the wake.

It is convenient to investigate the behaviour as r tends to infinity of certain integrals, taken over the sphere Σ, in which this exponential factor occurs. Consider first the integral

$$\mathrm{I}_1 = \iint_{\Sigma} e^{-k(r-x)} f(x, y, z)\, d\Sigma,$$

where

$$f = 0\left(\frac{1}{r^2}\right).$$

We may omit that part of the range of integration for which the integrand is exponentially small, *i.e.*, all points for which

$$\theta > \varepsilon = (kr)^{-\frac{1}{2}(1-\delta)}, \qquad \delta > 0.$$

Thus I_1 may be replaced by the integral

$$\mathrm{I}'_1 = r^2 \int_0^{2\pi} d\omega \int_0^{\varepsilon} e^{-k(r-x)} f(x, y, z) \sin\theta\, d\theta.$$

Now we have

$$|\mathrm{I}'_1| < \mathrm{A}r^2 \cdot \frac{1}{r^2} \int_0^{\varepsilon} \sin\theta\, d\theta$$

$$< \mathrm{A}\varepsilon^2,$$

where A is a numerical constant.

It follows that,

$$\lim_{r\to\infty} \mathrm{I}_1 = 0. \quad \ldots \ldots \ldots \ldots \quad (8.21)$$

The same result is evidently true if we omit the exponential factor in the integrand and take the integral over the wake only.

Proceeding in the same way, we obtain the following results :

$$\lim_{r\to\infty} \iint_{\Sigma} e^{-k(r-x)} f(x, y, z)\,(1 - \cos\theta)\, d\Sigma = 0 \quad \ldots \ldots \ldots \quad (8.22)$$

provided

$$f = 0\left(\frac{1}{r}\right).$$

$$\lim_{r\to\infty}\iint_{\Sigma} e^{-k(r-x)} f(x, y, z)\,(1-\cos\theta)^2\,d\Sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.23)$$

provided $f = 0\,(1)$.

$$\lim_{r\to\infty}\iint_{\Sigma} e^{-k(r-x)} f(x, y, z)\,\sin\theta\,d\Sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.24)$$

provided $f = 0\left(\frac{1}{r}\right)$.

$$\lim_{r\to\infty}\iint_{\Sigma} e^{-k(r-x)} f(x, y, z)\,\sin^3\theta\,d\Sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.25)$$

provided $f = 0\,(1)$.

$$\lim_{r\to\infty}\iint_{\Sigma} e^{-k(r-x)} f(x, y, z)\,(1-\cos\theta)\sin\theta\,d\Sigma = 0 \quad . \quad . \quad . \quad . \quad (8.26)$$

provided $f = 0\,(1)$.

$$\lim_{r\to\infty}\iint_{\Sigma} e^{-k(r-x)} f(x, y, z)\,(1-\cos\theta)\sin^3\theta\,d\Sigma = 0 \quad . \quad . \quad . \quad . \quad (8.27)$$

provided $f = 0\,(\mathrm{r})$.

In each case the result is still true if we omit the exponential factor and take the integral over the wake only.

Consider now the contribution to F of the velocities u_2, v_2, w_2 of the complementary function.

v_2 and w_2 are of the form

$$e^{-k(r-x)} f(x, y, z),$$

where

$$f = 0\left(\frac{1}{r}\right).$$

It follows from (8.24) that

$$\lim_{r\to\infty}\iint_{\Sigma} (m'v_2 + n'w_2)\,d\Sigma = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.31)$$

We now have to evaluate $\iint_{\Sigma} l'u_2\,d\Sigma$. It is only necessary to consider solutions such that u_2 is independent of ω, since in all other cases the integral with respect to ω vanishes. The only solutions of this form are those of type I for which

$$u_2 = (n+1)\alpha_n e^{kx}\,[G_n + G_{n+1}].$$

Now it has been shown that*

$$\lim_{r\to\infty}\iint_{\Sigma} e^{kx} l' G_n\,d\Sigma = \frac{2\pi}{k^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.32)$$

* Garstang, 'Proc. Roy. Soc.,' A, vol. 142, p. 502 (1933).

It follows that

$$\lim_{r\to\infty}\iint_{\Sigma} l'u_2\, d\Sigma = \frac{4\pi}{k^2}\sum_{n=0}^{\infty}(n+1)\,\alpha_n. \qquad (8.33)$$

Next consider the contribution to F of the irrotational solutions.

If

$$\phi = 0\left(\frac{1}{r^2}\right),$$

then

$$(u_1, v_1, w_1) = 0\left(\frac{1}{r^3}\right),$$

and it is clear that

$$\lim_{r\to\infty}\iint_{\Sigma}(l'u_1 + m'v_1 + n'w_1)\, d\Sigma = 0.$$

It follows that the only solution which contributes to F is that corresponding to the value

$$\phi = a_0 C_0 = \frac{a_0}{r}.$$

In this case we have

$$\iint_{\Sigma}(l'u_1 + m'v_1 + n'w_1)\, d\Sigma = -\iint_{\Sigma}\frac{\partial\phi}{\partial r}\, d\Sigma = 4\pi a_0. \qquad (8.4)$$

It is now necessary to consider at some length the contribution of the special solutions to F. This part of the work is complicated by the fact that the functions $R_n^m(\cos\theta)$ become infinite when $\theta = 0$.

Consider first the contribution of these solutions to $\iint_{\Sigma} l'u_1\, d\Sigma$. Corresponding to the value

$$\phi = \mathbf{C}_{n-1}^{\ n}, \qquad (8.50)$$

we have, making use of (6.43),

$$u_1 = -\frac{\partial \mathbf{C}_{n-1}^{\ n}}{\partial x} + u_3$$

$$= C_n^n + u_3.$$

The integral of the harmonic term always vanishes, since the integral with respect to ω is zero. It also vanishes by reason of its order of magnitude if $n > 1$.

Now from equations (7.61), (7.53), (7.55), and (7.56) we have, if $n > 1$,

$$u_3 = 2\frac{\partial^2}{\partial x\partial y}(e^{kx}\chi_{n-1})$$

$$= 2\frac{\partial}{\partial y}\left[2^{n-2}\frac{(n-2)!}{(2n-3)!}k^n e^{kx} G_{n-1}^{\ n-1}\right].$$

To deal with this expression we make use of certain special cases of equation (4.75), viz.

$$\frac{\partial G_{n-1}^{\ n-1}}{\partial y} = \frac{k}{2(2n-1)}[-G_n^{\ n} - (2n-2)(2n-3)\,G_{n-2}^{\ n-2} + 2G_n^{\ n-2}], \quad (n > 2) \quad (8.51)$$

$$\frac{\partial G_1^{\ 1}}{\partial y} = \frac{k}{6}[-G_2^{\ 2} - 2G_0 + 2G_2]. \quad \ldots \quad (8.52)$$

Also, if $n = 1$, we have from (7.72),

$$u_3 = -k^2 e^{kx} G_1^{\ 1}. \quad \ldots \quad (8.53)$$

Equations (8.51), (8.52), and (8.53) show that, in evaluating $\iint_\Sigma l' u_3 d\Sigma$, the integral with respect to ω vanishes in all cases except for the terms in $\frac{\partial G_1^{\ 1}}{\partial y}$, which are independent of ω, and the contributions of these terms cancel owing to the relation (8.32).

In the same way we can show that the contributions to $\iint_\Sigma l' u_1 d\Sigma$ of the solutions corresponding to the value

$$\phi = \mathbf{D}_{n-1}^{\ n}$$

also vanish in the limit, and we have therefore

$$\lim_{r\to\infty} \iint_\Sigma l' u_1\, d\Sigma = 0. \quad \ldots \quad (8.54)$$

We now turn to the contribution of the special solutions to

$$\iint_\Sigma m' v_1\, d\Sigma.$$

Consider first the solution corresponding to the value

$$\phi = \mathbf{C}_0^{\ 1},$$

v_1 being given by (7.73) and (7.74).

The expression $e^{kx}\,G_0$, being a continuous solution of (3.4), is one of the terms which appear in v_2, and it follows from (8.31) that the integral of this term vanishes in the limit.

With regard to the other terms, we see from (7.74) that the integral over the part of Σ outside the wake certainly vanishes, since the integral with respect to ω is zero.

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Thus we have only to consider the behaviour within the wake of the terms which become infinite for $\theta = 0$. Using equation (7.73), and writing for shortness

$$k(r-x) = \xi,$$

we find

$$\frac{\partial}{\partial y}(\mathbf{C}_0^1 - ke^{kx}\mathbf{G}_0^1) = \frac{k(r^2-y^2)}{r^3}\left[-1+\frac{\xi}{2!}-\frac{\xi^2}{3!}+\ldots\right] + \frac{k^2y^2}{2r^2} + \frac{k^2y^2\xi}{r}\left[-\frac{2}{3!}+\frac{3\xi}{4!}-\frac{4\xi^2}{5!}+\ldots\right]$$
$$= f_1(x,y,z) + \sin^2\theta\, f_2 + \sin^2\theta\,(1-\cos\theta)\, f_3,$$

where $f_1 = 0\left(\frac{1}{r}\right)$, $f_1 = 0(1)$, $f_3 = 0(r)$.

Thus, making use of equations (8.24), (8.25), and (8.27), we see that

$$\lim_{r\to\infty}\iint_\Sigma m'v_1\, d\Sigma = 0.$$

Considering next the solution corresponding to the value

$$\phi = \mathbf{C}_1^2,$$

we have, from (7.62),

$$v_1 = -\frac{\partial \mathbf{C}_1^2}{\partial y} - \frac{\partial}{\partial y}(e^{kx}\psi_2) - 4k^3e^{kx}\,\mathrm{G}_1^1.$$

With the help of the differentiation formulae given in §§4 and 6, we find

$$v_1 = 2\frac{\partial^2}{\partial y^2}(\mathbf{C}_0^1 - ke^{kx}\mathbf{G}_0^1) + g_1(x,y,z) + g_2(x,y,z), \quad \ldots \quad (8.6)$$

where g_1 is a continuous harmonic function of order $\frac{1}{r^3}$ and g_2 is a continuous solution of (3.4). It is only necessary to consider the first term on the right-hand side of (8.6). The integral of this term over the part of Σ outside the wake vanishes in the limit, since $\frac{\partial^2\mathbf{C}_0^1}{\partial y^2}$ is a harmonic function of order $\frac{1}{r^3}$ and $\frac{\partial^2}{\partial y^2}(e^{kx}\mathbf{G}_0^1)$ is exponentially small. Within the wake, more careful investigation is required, and we find

$$\frac{\partial^2}{\partial y^2}(\mathbf{C}_0^1 - ke^{kx}\mathbf{G}_0^1) = k\left[-1+\frac{\xi}{2!}-\frac{\xi^2}{3!}+\ldots\right]\frac{\partial}{\partial y}\left(\frac{r^2-y^2}{r^3}\right) + \frac{3k^2y(r^2-y^2)}{r^4}\left[\frac{1}{2!}-\frac{2\xi}{3!}+\frac{3\xi^2}{4!}-\ldots\right] + \frac{k^3y^3}{r^3}\left[-\frac{2}{3!}+\frac{3.2\xi}{4!}-\frac{4.3\xi^2}{5!}+\ldots\right]$$
$$= f_1(x,y,z) + \sin\theta\, f_2(x,y,z) + \sin^3\theta\, f_3(x,y,z),$$

where

$$f_1 = 0\left(\frac{1}{r^2}\right), \qquad f_2 = 0\left(\frac{1}{r}\right), \qquad f_3 = 0(1).$$

It now follows from equations (8.21), (8.24), and (8.25) that

$$\lim_{r\to\infty} \iint_{\Sigma} m' v_1 \, d\Sigma = 0.$$

It is easily seen that the value of v_1 corresponding to the value $\mathbf{C}_{n-1}{}^{n}$ of ϕ can be expressed in the form

$$v_1 = \mathrm{A} \frac{\partial^n}{\partial y^n} (\mathbf{C}_0{}^1 - ke^{kx}\mathbf{G}_0{}^1) + g_1 (x, y, z) + g_2 (x, y, z),$$

where A is a numerical constant, and g_1 and g_2 are of the same form as the functions g_1 and g_2 in equation (8.6). Also, further differentiation of $\mathbf{C}_0{}^1 - ke^{kx}\mathbf{G}_0{}^1$ with respect to y brings in an extra factor $1/r$ in some terms, and an extra factor $\sin\,\theta$ in the others, either of which has the effect of making the corresponding integrals tend to zero more rapidly.

Thus all the contributions to $\iint_{\Sigma} m' v_1 \, d\Sigma$ of the series of special solutions given by (7.23) vanish, and, similarly, the contributions of the series of special solutions given by (7.25) also vanish.

In the same way we can show that the special solutions contribute nothing to $\iint_{\Sigma} n' w_1 \, d\Sigma$.

Thus it appears that the special solutions give no total flow over Σ in the limit.

Collecting our results, it follows from (8.12), (8.33), and (8.4) that

$$a_0 = -\frac{1}{k^2} \sum_{n=0}^{\infty} (n+1)\,\alpha_n. \qquad (8.7)$$

Now the coefficients a_0 and α_n are associated with solutions of equations (3.1) and (3.2) which possess axial symmetry. It follows that equation (8.7) is exactly the same, allowing for differences in notation, as the equations given by GOLDSTEIN* and DAHL,† although these authors, in the papers referred to, confine themselves to the case of axial symmetry.

The expression $4\pi a_0$ represents a flow uniformly distributed over Σ, this flow being outwards if a_0 is positive. On the other hand, since the integrand on the left-hand side of (8.32) is exponentially small except in the wake, the expression $\frac{4\pi}{k^2} \sum_{n=0}^{\infty} (n+1)\,\alpha_n$ represents a flow which is confined to the wake and which is inwards if $\sum_{n=0}^{\infty} (n+1)\,\alpha_n$ is negative. We shall find later when discussing the drag that this latter condition is always satisfied, and therefore $4\pi a_0$ represents an outward flow. Anticipating this result and denoting the outward flow over Σ by E, we have

$$\mathrm{E} = 4\pi a_0 = -\frac{4\pi}{k^2} \sum_{n=0}^{\infty} (n+1)\,\alpha_n. \qquad (8.8)$$

* 'Proc. Roy. Soc.,' A, vol. 123, p. 232, equation (54) (1929).

† 'Ark. Mat. Astr. Fys.,' vol. 21, No. 5, p. 19, equation (29) (1928).

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9—The Drag

Let (X, Y, Z) denote the forces exerted by the fluid on the solid, and (X′, Y′, Z′) the forces exerted by the fluid outside Σ on the fluid inside.

The rate of transfer of momentum across Σ, parallel to the axis of x, is readily shown to be

$$\iint_{\Sigma} \rho\,(U+u)\,\{l'\,(U+u)+m'v+n'w\}\,d\Sigma = M_x, \quad \text{say.} \quad . \quad . \quad (9.11)$$

This must be equal to the total external force applied to the fluid, in the same direction, across the boundaries S and Σ, whence we obtain, as the equation giving the drag,

$$X' - X = \iint_{\Sigma} \rho\,(U+u)\,\{l'\,(U+u)+m'v+n'w\}\,d\Sigma. \quad . \quad . \quad . \quad (9.12)$$

Now X′ is given by

$$X' = \iint_{\Sigma} (l'\widehat{xx} + m'\widehat{xy} + n'\,\widehat{xz})\,d\Sigma$$

$$= \iint_{\Sigma} \left\{ -\,l'p + \rho\nu\left[2l'\,\frac{\partial u}{\partial x} + m'\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) + n'\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]\right\} d\Sigma.$$

If (ξ, η, ζ) is the vorticity, then

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

and we have

$$X' = \iint_{\Sigma} \{-\,l'p - \rho\nu\,(m'\zeta - n'\eta)\}\,d\Sigma + 2\rho\nu \iint_{\Sigma}\left(l'\,\frac{\partial u}{\partial x} + m'\,\frac{\partial v}{\partial x} + n'\,\frac{\partial w}{\partial x}\right) d\Sigma.$$

It has been shown by Goldstein that the last integral vanishes identically,* and that the integrals of the terms in η and ζ vanish in the limit.† We thus have

$$X' = -\iint_{\Sigma} l'p\,d\Sigma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.2)$$

Again,

$$M_x = \iint_{\Sigma} \rho l'U^2\,d\Sigma + \iint_{\Sigma} \rho U\,(2l'u + m'v + n'w)\,d\Sigma$$
$$+ \iint_{\Sigma} \rho u\,(l'u + m'v + n'w)\,d\Sigma. \quad . \quad . \quad (9.31)$$

The first integral vanishes identically. Hence, making use of (8·12), we have

$$M_x = \rho U \iint_{\Sigma} l'u\,d\Sigma + \rho \iint_{\Sigma} u\,(l'u + m'v + n'w)\,d\Sigma. \quad . \quad . \quad . \quad . \quad (9.32)$$

* 'Proc. Roy. Soc.,' A, vol. 123, p. 221 (1929).

† *Ibid.*, vol. 131, p. 205 (1931).

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GOLDSTEIN* has given in his first paper a proof that the last integral in (9.32) tends to zero. The proof only applies, however, to the incomplete solution of the equations given in that paper. In order to obtain a general proof, we make use of the discussion of the integral $\iint_{\Sigma} (l'u + m'v + n'w)\, d\Sigma$ given in §8. It was found that most of the terms in this integral vanished in the limit on account of their order of magnitude. In all these cases, multiplication of the integrand by u will cause the integral to tend to zero still more rapidly. Further, in those cases which formerly depended on the fact that the integral with respect to ω vanished, and those which actually led to a finite result, it is easily seen, with the help of equations (8.21) to (8.27), that multiplication by u will cause the integral to vanish on account of its order of magnitude. Thus we have

$$\lim_{r\to\infty} \iint_{\Sigma} u\,(l'u + m'v + n'w)\, d\Sigma = 0,$$

whence

$$\mathrm{M}_x = \rho \mathrm{U} \iint_{\Sigma} l'u\, d\Sigma. \qquad (9.33)$$

We now have from (9.12), (9.2), and (9.33),

$$\mathrm{X} = -\iint_{\Sigma} l'\,(p + \rho \mathrm{U} u)\, d\Sigma.$$

But

$$p = \rho \mathrm{U} \frac{\partial \phi}{\partial x},$$

and

$$u = -\frac{\partial \phi}{\partial x} + u_3 + u_2,$$

from which we obtain

$$\mathrm{X} = -\rho \mathrm{U} \iint_{\Sigma} l'\,(u_3 + u_2)\, d\Sigma. \qquad (9.41)$$

Making use of (8.33) and (8.54), we have

$$\mathrm{X} = -\frac{4\pi\rho\mathrm{U}}{k^2} \sum_{n=0}^{\infty} (n+1)\,\alpha_n. \qquad (9.42)$$

From physical considerations X must be positive, and therefore $\sum_{n=0}^{\infty} (n+1)\,\alpha_n$ must be negative, which is the result that was assumed in §8. We have from (9.42) and (8.8),

$$\mathrm{X} = 4\pi\rho\mathrm{U} a_0 \qquad (9.5)$$

$$= \rho \mathrm{U E}. \qquad (9.6)$$

* 'Proc. Roy. Soc.,' A, vol. 123, p. 216 (1929).

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The value of the drag depends only on those solutions of (3.1) and (3.2) which possess axial symmetry. For this reason, as with the condition for no total flow, equation (9.5) agrees with the equations given by GOLDSTEIN* and DAHL.† DAHL's results are, in the first instance, only valid for small values of the Reynolds number, since he calculates the drag by means of integrals over the solid; they can, however, be readily extended to the general case by using integrals at infinity instead, since the discussion relates to flow past a fixed body.‡

Equation (9.6) has also been given by GOLDSTEIN§ for unrestricted three-dimensional flow. The same equation was obtained by FILON|| for two-dimensional flow.

10—THE LIFT

We now turn to the calculation of the lift on the body in the direction of the axis of y. The rate of transfer of momentum across Σ, parallel to this axis, is given by

$$M_y = \iint_\Sigma \rho v \{l' (U + u) + m'v + n'w\} \, d\Sigma.$$

The equation for the lift is

$$Y' - Y = M_y. \qquad (10.1)$$

Now Y' is given by

$$\begin{aligned} Y' &= \iint_\Sigma (l'\widehat{xy} + m'\widehat{yy} + n'\widehat{yz}) \, d\Sigma \\ &= \iint_\Sigma \left\{\rho\nu l' \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + m' \left(-p + 2\rho\nu \frac{\partial v}{\partial y}\right) + \rho\nu n' \left(\frac{\partial w}{\partial y} + \frac{\partial v}{dz}\right)\right\} d\Sigma \\ &= \iint_\Sigma [-m'p + \rho\nu (l'\zeta - n'\xi)] \, d\Sigma + \iint_\Sigma 2\rho\nu \left(l' \frac{\partial u}{\partial y} + m' \frac{\partial v}{\partial y} + n' \frac{\partial w}{\partial y}\right) d\Sigma. \end{aligned}$$

As for the drag, the last integral vanishes identically, and the integrals of the terms in ξ and ζ vanish in the limit. We thus have

$$\begin{aligned} Y' &= -\iint_\Sigma m'p \, d\Sigma. \\ &= -\rho U \iint_\Sigma m' \frac{\partial \phi}{\partial x} \, d\Sigma. \qquad (10.2) \end{aligned}$$

Also

$$M_y = \iint_\Sigma \rho U l'v \, d\Sigma + \iint_\Sigma \rho v(l'u + m'v + n'w) d\Sigma.$$

* 'Proc. Roy. Soc.,' A, vol. 123, p. 227, equation (20) (1929).
† 'Ark. Mat. Astro. Fys.,' vol. 21, No. 5, p. 19, equation (30) (1928).
‡ GARSTANG, 'Proc. Roy. Soc.,' A, vol. 142, p. 491 (1933), §7.
§ 'Proc. Roy. Soc.,' A, vol. 131, p. 198 (1931).
|| *Ibid.*, vol. 113, p. 7 (1926).

As for the drag, the last integral vanishes in the limit, and we have

$$M_y = \rho U \iint_\Sigma l' v \, d\Sigma. \quad \ldots \ldots \ldots \ldots \quad (10.3)$$

Thus from (10.1), (10.2), and (10.3), we obtain

$$Y = -\rho U \iint_\Sigma \left[l' \left(-\frac{\partial \phi}{\partial y} + v_3 + v_4 + v_2 \right) + m' \frac{\partial \phi}{\partial x} \right] d\Sigma. \quad \ldots \quad (10.41)$$

Now since all the terms in u_3 are solutions of (3.4), which remain finite for $\theta = 0$, it follows from (8.24) that

$$\lim_{\to\infty} \iint_\Sigma m' u_3 \, d\Sigma = 0.$$

Thus (10.41) may be written

$$Y = -\rho U \iint_\Sigma \left[l' \left(-\frac{\partial \phi}{\partial y} + v_3 \right) - m' \left(-\frac{\partial \phi}{\partial x} + u_3 \right) \right] d\Sigma - \rho U \iint_\Sigma l'(v_4 + v_2) d\Sigma.$$

It has been remarked in §7 that u_3, v_3, w_3 are in all cases the gradient of a single-valued function ; hence, making use of a result given by Goldstein,* we see that the first integral vanishes identically, whence

$$Y = -\rho U \iint_\Sigma l'(v_4 + v_2) \, d\Sigma. \quad \ldots \ldots \quad (10.42)$$

Consider now the value of

$$\iint_\Sigma l' v_2 \, d\Sigma.$$

It is only necessary to consider solutions such that v_2 is independent of ω, since in all other cases the integral with respect to ω vanishes. The only solutions of this form are those of type II for which

$$v_2 = (n+1) \beta_n e^{kx} [G_n - G_{n+1}].$$

It follows at once from (8.32) that

$$\lim_{r\to\infty} \iint_\Sigma l' v_2 d\Sigma = 0.$$

Thus it appears that the lift is given by

$$Y = -\rho U \iint_\Sigma l' v_4 \, d\Sigma. \quad \ldots \ldots \ldots \ldots \quad (10.5)$$

Corresponding to the value $\mathbf{C}^n{}_{n-1}$ of ϕ, we have from (7.62), if $n > 1$,

$$v_4 = -\frac{(n-2)!}{(2n-3)!} 2^n k^{n+1} e^{kx} G_{n-1}{}^{n-1}.$$

* 'Proc. Roy. Soc.,' A, vol. 123, p. 221 (1929).

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This contributes to Y a term

$$\rho U \frac{(n-2)\,!}{(2n-3)\,!} 2^n k^{n+1} \iint_{\Sigma} l' e^{kx} G_{n-1}{}^{n-1}\, d\Sigma,$$

which vanishes since the integral with respect to ω is zero.

Corresponding to the value $a_0{}^1\, \mathbf{C}_0{}^1$ of ϕ, we have from (7.73),

$$v_4 = -\, 2a_0{}^1\, k^2 e^{kx} G_0,$$

which leads to

$$Y = 2\, a_0{}^1\, k^2 \rho U \iint_{\Sigma} l' e^{kx}\, G_0 d\Sigma.$$

It follows from (8.32) that

$$Y = 4\pi\rho U a_0{}^1. \quad \ldots \ldots \ldots \ldots \quad (10.6)$$

Similarly, corresponding to the value $b_0{}^1 \mathbf{D}_0{}^1$ of ϕ, there is a lift in the direction of the axis of z given by

$$Z = 4\pi\rho U b_0{}^1. \quad (10.7)$$

The physical significance of the constants $a_0{}^1$, $b_0{}^1$ is discussed in the next section. It may be remarked here that there is an evident symmetry about the equations (9.5), (10.6), and (10.7) ; also these equations show that the forces on the body are associated with the three particular integrals corresponding to the values of p which are spherical harmonics of degree $-\,2$.

The discussion of the lift which has been given by Bateman* is incorrect. Starting from values of the velocities which satisfy Oseen's equations, Bateman obtains the formula

$$L = \rho U \iint \left(l \frac{\partial \phi}{\partial y} - m \frac{\partial \phi}{\partial x} \right) dS, \quad \ldots \ldots \quad (10.81)$$

where L denotes the lift in the direction of the y-axis and the integral is to be taken over the surface of the solid. ϕ satisfies (7.13) and the pressure is given by (7.12). Also u, v, w are given by (7.11), if we omit certain rotational velocities, included in those discussed in §4, which are found not to affect the value of L. Bateman then says :

" If, when z is kept constant, ϕ is a single-valued function of x and y, as in the case of the sphere, the lift L vanishes, but if it increases by $T(c)$ when a point in the plane $z = c$ describes a closed curve in which this plane cuts the surface of the body, then the formula for the lift is

$$L = \rho U \int T(z) dz.$$

In particular, if ϕ contains a term of type

$$f(z) \tan^{-1} (y/x), \quad \ldots \ldots \ldots \quad (10.82)$$

* Bateman, Dryden, and Murnaghan, 'Bull. Nat. Res. Coun.,' No. 84, §7.6, p. 317 (1932).

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then this term contributes to the lift an amount

$$2\pi\rho U \int f(z)\, dz."$$

Now the values of ϕ which actually occur in the solution of OSEEN's equations, *i.e.*, (3.1) and (3.2), are given by equations (7.21) to (7.25), and they are all single-valued functions. Apart from this, however, it is easily shown that no such value of ϕ as that given by (10.82) can possibly occur. For

$$\nabla^2 f(z) \tan^{-1}(y/x) = \frac{d^2 f}{dz^2} \tan^{-1}(y/x),$$

from which we obtain

$$\frac{d^2 f}{dz^2} = 0,$$

$$\text{and } \phi = (A + Bz) \tan^{-1}(y/x).$$

The pressure is therefore given by

$$p = -\rho U (A + Bz)\, y/(x^2 + y^2),$$

and this expression does not tend to zero in all directions at a great distance from the solid. The corresponding irrotational velocities are also unsuitable for the same reason. We might try to get over the latter difficulty by adding a suitable solution of (3.4). It can be shown that this is not possible, but it is unnecessary to give the proof here, since the fact that the pressure does not satisfy the necessary conditions is sufficient to render the value of ϕ given by (10.82) inadmissible. Thus the formula (10.81) does not really give a lift at all.

11—THE CIRCULATION IN THE WAKE

It is well known that in practice the lifting force on a body is always associated with a system of trailing vortices in the wake. We therefore proceed to investigate the distribution of vorticity at a great distance from the body in the present theoretical solution.

Consider the surface integral of the normal component of vorticity over the half of the sphere Σ for which z is positive. If this hemisphere is denoted by Σ', the integral is given by

$$\iint_{\Sigma'} (l'\xi + m'\eta + n'\zeta)\, d\Sigma. \qquad (11.11)$$

The vorticity involves only continuous solutions of equation (3.4), and therefore ξ, η, ζ are exponentially small except in the wake. It follows that the integral (11.11) may be replaced by an integral over the half of the wake for which z is positive. Now, making use of STOKES's theorem, we see that the latter integral is equal to the circulation I in a circuit C_1 enclosing the same half of the wake ; the sense of description of this circuit being that of a right-handed rotation about the outward-drawn normal to Σ.

It is convenient now to anticipate the results obtained below. We shall find that for the solution giving the lift in the direction of the y-axis, the circulation in the circuit C_1 tends to zero as $r^{-\frac{1}{2}}$, but for all other solutions it tends to zero more rapidly than $r^{-\frac{1}{2}}$. We therefore proceed to evaluate the expression

$$\lim_{r\to\infty} r^{\frac{1}{2}} \iint_{\Sigma'} (l'\xi + m'\eta + n'\zeta)\, d\Sigma = \mathrm{W}, \text{ say.} \quad \ldots\ldots \quad (11.12)$$

As a preliminary to this, we shall investigate the behaviour as r tends to infinity of certain integrals taken over the hemisphere Σ'. We start with the expression

$$\mathrm{U}_n^m = r^{\frac{1}{2}} \iint_{\Sigma'} l' e^{kx}\, \mathrm{H}_n^m\, d\Sigma.$$

The integrand is of the form

$$e^{-k(r-x)} f(x, y, z) \sin^m\theta, \text{ where } f = 0\left(\frac{1}{r}\right),$$

and it follows from (8.24) that

$$\lim_{r\to\infty} r^{\frac{1}{2}} \iint_{\Sigma'} e^{-k(r-x)} f(x, y, z) \sin^m\theta\, d\Sigma = 0$$

provided

$$f = 0\left(\frac{1}{r}\right) \text{ and } m > 1.$$

Hence

$$\lim_{r\to\infty} \mathrm{U}_n^m = 0, \qquad \text{provided } m > 1. \quad \ldots\ldots \quad (11.20)$$

Similarly it can be shown that if

$$\mathrm{V}_n^m = r^{\frac{1}{2}} \iint_{\Sigma'} m' e^{kx}\, \mathrm{H}_n^m\, d\Sigma,$$

then

$$\lim_{r\to\infty} \mathrm{V}_n^m = 0 \quad \ldots\ldots\ldots\ldots \quad (11.21)$$

for all values of m. Equation (11.21) remains true if in the integrand we substitute n' for m' or G_n^m (but not G_m) for H_n^m.

Consider now the expression

$$\mathrm{U}_n^1 = r^{\frac{1}{2}} \iint_{\Sigma'} l' e^{kx}\, \mathrm{H}_n^1\, d\Sigma.$$

Retaining only the term of greatest order in r, we have

$$\mathrm{U}_n^1 = \frac{r^{\frac{3}{2}}}{k} \int_0^{\pi} \sin\omega\, d\omega \int_0^{\pi} e^{-kr(1-\cos\theta)} \sin^2\theta\, \frac{d\mathrm{P}_n(\cos\theta)}{d(\cos\theta)}\, d\theta$$

$$= \frac{2r^{\frac{3}{2}}}{k} \int_0^{\pi} e^{-kr(1-\cos\theta)} \sin^2\theta\, \frac{d\mathrm{P}_n(\cos\theta)}{d(\cos\theta)}\, d\theta.$$

This may be replaced by an integral taken over the wake only, and in this region we may replace $1 - \cos\theta$ by $\frac{1}{2}\theta^2$, $\sin\theta$ by θ, and $\frac{dP_n(\cos\theta)}{d(\cos\theta)}$ by its value when $\theta = 0$.

Writing RODRIGUE's formula in the form

$$P_n(\mu) = \frac{1}{2^n\, n!}\frac{d^n}{d\mu^n}\{(\mu+1)^n(\mu-1)^n\},$$

and using LEIBNIZ's theorem, we find that

$$\left[\frac{dP_n(\mu)}{d\mu}\right]_{\mu\,=\,1} = \frac{n(n+1)}{2},$$

whence

$$U_n^1 = \frac{n(n+1)r^{\frac{3}{2}}}{k}\int_0^{\epsilon} e^{-\frac{1}{2}kr\,\theta^2}\,\theta^2\,d\theta.$$

Denote now by J the expression

$$\frac{r^{\frac{3}{2}}}{k}\int_0 e^{-\frac{1}{2}kr\,\theta^2}\,\theta^2\,d\theta.$$

If we put $\frac{1}{2}kr = R^2$, $R\theta = \alpha$, we have

$$J = \frac{2}{k^2}\left(\frac{2}{k}\right)^{\frac{1}{2}}\int_0^{R\epsilon} e^{-\alpha^2}\,\alpha^2\,d\alpha,$$

whence

$$\lim_{r\to\infty} J = \frac{2}{k^2}\left(\frac{2}{k}\right)^{\frac{1}{2}}\int_0^{\infty} e^{-\alpha^2}\,\alpha^2\,d\alpha = \frac{1}{2k^2}\left(\frac{2\pi}{k}\right)^{\frac{1}{2}} = \bar{J},\ \text{say.} \quad \ldots\ldots \quad (11.22)$$

Thus

$$\lim_{r\to\infty} U_n^1 = n(n+1)\,\bar{J}. \quad \ldots\ldots\ldots\ldots \quad (11.23)$$

Since $\bar{J}$ is finite, it follows from (11.23) that

$$\lim_{r\to\infty}\ [(n+1)(n+2)\,U_{n-1}^1 - (n-1)\,nU_{n+1}^1] = 0, \quad \ldots \quad (11.24)$$

and

$$\lim_{r\to\infty}\left[\frac{(U_{n-1}^1 - U_{n+1}^1)}{2n+1} - \frac{(U_n^1 - U_{n+2}^1)}{2n+3}\right] = 0. \quad \ldots \quad (11.25)$$

We now turn to the expression

$$V_n = r^{\frac{1}{2}}\iint_{\Sigma'} m' e^{kx}\, G_n\, d\Sigma.$$

Retaining only the term of greatest order in r, and making use of equation (4.30) we have

$$V_n = \frac{r^{\frac{3}{2}}}{(2n+1)\,k}\int_0^{\pi}\sin\omega\,d\omega\int_{\theta}^{\pi} e^{-kr(1-\cos\theta)}\sin\theta\,(P_{n+1}^1 - P_{n-1}^1)\,d\theta$$

$$= \frac{2r^{\frac{3}{2}}}{(2n+1)\,k}\int_0^{\pi} e^{-kr(1-\cos\theta)}\sin^2\theta\left[\frac{dP_{n+1}(\cos\theta)}{d(\cos\theta)} - \frac{dP_{n-1}(\cos\theta)}{d(\cos\theta)}\right]d\theta.$$

This integral may be evaluated by the method used for U_n^1, and we find that

$$\lim_{r \to \infty} V_n = 2\bar{J}. \qquad (11.26)$$

It follows from (11.26) that

$$\lim_{r \to \infty} (V_{n-1} - V_{n+1}) = 0, \qquad (11.27)$$

and

$$\lim_{r \to \infty} [nV_{n-1} - (2n+1) V_n + (n+1) V_{n+1}] = 0. \qquad (11.28)$$

Similarly it can be shown that if

$$W_n = r^{\frac{1}{2}} \iint_{\Sigma'} n' e^{kx} G_n \, d\Sigma,$$

then

$$\lim_{r \to \infty} W_n = 2\bar{J}. \qquad (11.31)$$

It follows from (11·31) that

$$\lim_{r \to \infty} (W_0 - W_1) = 0. \qquad (11.32)$$

Finally we consider the integral

$$L'_n = \iint_{\Sigma'} l' e^{kx} G_n \, d\Sigma.$$

It has been shown that* if

$$L_n = \iint_{\Sigma} l' e^{kx} G_n \, d\Sigma,$$

then

$$\lim_{r \to \infty} L_n = \frac{2\pi}{k^2}.$$

It easily follows from the investigation referred to that a second approximation to L_n when r is large is given by

$$L_n = \frac{2\pi}{k^2} + \frac{2A_n}{r},$$

where A_n is a numerical constant. Owing to the axial symmetry of the integrand, we have, to the same order of approximation,

$$L'_n = \tfrac{1}{2} L_n = \frac{\pi}{k^2} + \frac{A_n}{r}. \qquad (11.33)$$

It follows from (11.33) that

$$\lim_{r \to \infty} r^{\frac{1}{2}} (L'_{n-1} - L'_{n+1}) = 0. \qquad (11.34)$$

* GARSTANG, 'Proc. Roy. Soc.,' A, vol. 142, p. 502 (1933).

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Consider now the contribution to W of the special solution associated with the lift in the direction of the y-axis. Corresponding to the value $a_0{}^1\,\mathbf{C}_0{}^1$ of ϕ, we find, from equations (7.71), (7.73), (7.75), (4.71), and (4.77), that

$$\xi = -\,2a_0{}^1\,k^3e^{kx}\,\mathrm{H}_1{}^1,\ \eta = 0,$$

$$\zeta = -\,2a_0{}^1\,k^3e^{kx}\,(\mathrm{G}_0 - \mathrm{G}_1).$$

It follows from (11.22) and (11.23) that

$$\lim_{r\to\infty} r^{\frac{1}{2}} \iint_{\Sigma'} l'\xi\, d\Sigma = -\,2\,(2\pi k)^{\frac{1}{2}}\,a_0{}^1.$$

Also from (11.32) we have

$$\lim_{r\to\infty} r^{\frac{1}{2}} \iint_{\Sigma'} n'\zeta\, d\Sigma = 0.$$

Thus, considering this solution alone, we have

$$\lim_{r\to\infty} r^{\frac{1}{2}}\,\mathrm{I} = \mathrm{W} = -2\,(2\pi k)^{\frac{1}{2}}\,a_0{}^1. \qquad (11.40)$$

Since

$$\xi\,(x, y, -z) = -\,\xi\,(x, y, z),$$

there is evidently a circulation of the same magnitude I, but in the opposite sense, in a circuit C_2 enclosing the half of the wake for which z is negative.

Next we consider the contribution to W of the special solution corresponding to the value

$$\phi = \mathbf{C}_{n-1}{}^n, \quad \text{where } n > 1. \qquad (11.41)$$

It follows from equations (7.61) to (7.63) that, omitting a numerical factor, the vorticity depends on the term

$$v = e^{kx}\,\mathrm{G}_{n-1}{}^{n-1}.$$

With the help of equation (4.78) we find that ξ involves the functions $e^{kx}\,\mathrm{H}_n{}^n$, $e^{kx}\,\mathrm{H}_{n-2}{}^{n-2}$, and $e^{kx}\,\mathrm{H}_n{}^{n-2}$. On referring to (11.20) we see that $r^{\frac{1}{2}} \iint_{\Sigma'} l'\xi\, d\Sigma$ certainly tends to zero in all cases except where ξ includes terms of the form $\mathrm{H}_n{}^1$. These occur only if $n = 3$, when we find

$$\xi = -\,\frac{ke^{kx}}{10}\,(12\mathrm{H}_1{}^1 - 2\,\mathrm{H}_3{}^1) + \text{a term in } e^{kx}\,\mathrm{H}_3{}^3.$$

It follows from (11.24) that $r^{\frac{1}{2}} \iint_{\Sigma'} l'\xi\, d\Sigma$ tends to zero in this case also.

Again, we find that ζ involves the functions $e^{kx}\,G_{n-1}^{\;n-1}$, $e^{kx}\,G_n^{\;n-1}$, and it follows from (11·21) that $r^{\frac{1}{2}}\iint_{\Sigma'} n'\zeta\,d\Sigma$ vanishes in the limit, since by hypothesis $n > 1$. Thus the special solution corresponding to the value of ϕ given by (11.41) contributes nothing to W.

In the same way we can show that the special solution corresponding to the value

$$\phi = \mathbf{D}_{n-1}^{\;n}$$

contributes nothing to W provided $n > 1$. The case $n = 1$ is the solution associated with the lift in the direction of the z-axis. This solution will evidently give equal and opposite circulations I′, say, tending to zero as $r^{-\frac{1}{2}}$, in circuits enclosing the portions of the wake for which y is respectively positive and negative. But the solution contributes nothing to W, for corresponding to the value $b_0^{\;1}\,\mathbf{D}_0^{\;1}$ of ϕ, we find, from equations (7.81), (7.83), (7.85), (4.71), and (4.74) that

$$\xi = 2b_0^{\;1}\,k^3\,e^{kx}\,G_1^{\;1},$$

$$\eta = 2b_0^{\;1}\,k^3\,e^{kx}\,[G_0 - G_1],\ \zeta = 0.$$

Now $G_1^{\;1}$ and m' each contain the factor $\cos\omega$, and since

$$\int_0^{\pi}\cos\omega\,d\omega = 0,$$

it follows that $r^{\frac{1}{2}}\iint_{\Sigma'} l'\xi\,d\Sigma$ and $r^{\frac{1}{2}}\iint_{\Sigma'} m'\eta\,d\Sigma$ both vanish identically.

We now have to investigate the contribution to W of the various solutions of the complementary function. Consider first the value of $r^{\frac{1}{2}}\iint_{\Sigma'} l'\xi\,d\Sigma$. For type I solutions we find, with the help of equations (4.81), (4.76), and (4.78) that ξ involves the functions $e^{kx}H_{n+r}^{\;m}$, $r = -1, 0, 1, 2$. On referring to (11.20) we see that the integrals of these terms vanish if $m > 1$.

If $m = 1$, we have

$$\xi = \frac{\alpha_n^{\;1}ke^{kx}}{2n+1}\,[(n+2)\,(n+1)\,H_{n-1}^{\;1} - (n-1)\,nH_{n+1}^{\;1}] \\ - \frac{\alpha_n^{\;1}ke^{kx}}{2n+3}[(n+3)\,(n+2)\,H_n^{\;1} - n\,(n+1)\,H_{n+2}^{\;1}],$$

and it follows from (11.24) that $r^{\frac{1}{2}}\iint_{\Sigma'} l'\xi\,d\Sigma$ vanishes in the limit. If $m = 0$, we have $\xi = 0$.

Now consider the type II solutions. In this case we find that ξ involves the functions $e^{kx}H_{n+r}{}^{m+1}$, $r = -1, 0, 1, 2$, and making use of (11.20), we see that $r^{\frac{1}{2}}\iint_{\Sigma'} l'\xi\, d\Sigma$ tends to zero if $m > 0$. If $m = 0$, we have

$$\xi = -\frac{(n+1)\beta_n k e^{kx}}{2n+1}[H_{n-1}{}^1 - H_{n+1}{}^1] + \frac{(n+1)\beta_n k e^{kx}}{2n+3}[H_n{}^1 - H_{n+2}{}^1],$$

and it follows from (11.25) that $r^{\frac{1}{2}}\iint_{\Sigma'} l'\xi\, d\Sigma$ vanishes in the limit.

Turning now to the type III solutions, we find that ξ involves the functions $e^{kx}G_{n+r}{}^m$, $r = -1, 0, 1, 2$. Since $G_n{}^m$ contains the factor $\cos m\omega$ and

$$\int_0^{\pi} \cos m\omega\, d\omega = 0,$$

it follows that $r^{\frac{1}{2}}\iint_{\Sigma'} l'\xi\, d\Sigma$ vanishes identically unless $m = 0$. In this case we have

$$\xi = \frac{n(n+1)\gamma_n k e^{kx}}{2n+1}[-G_{n-1} + G_{n+1}] - \frac{(n+1)(n+2)\gamma_n k e^{kx}}{2n+3}[-G_n + G_{n+2}],$$

and it follows from (11.34) that $r^{\frac{1}{2}}\iint_{\Sigma'} l'\xi\, d\Sigma$ vanishes in the limit.

Finally, for the type IV solutions, we find that ξ involves the functions $e^{kx}G_{n+r}{}^{m+1}$, $r = -1, 0, 1, 2$, and here $r^{\frac{1}{2}}\iint_{\Sigma'} l'\zeta\, d\Sigma$ vanishes identically for all values of m.

Consider next the contribution of the type I solutions to $r^{\frac{1}{2}}\iint_{\Sigma'} m'\eta\, d\Sigma$. We find that η involves the functions $e^{kx}H_n{}^m$, but not the functions $e^{kx}G_n{}^m$, and it follows from (11.21) that $r^{\frac{1}{2}}\iint_{\Sigma'} m'\eta\, d\Sigma$ tends to zero for all values of m. For the same reason the contribution of the type II solutions to $r^{\frac{1}{2}}\iint_{\Sigma'} m'\eta\, d\Sigma$ and the contributions of the type III and type IV solutions to $r^{\frac{1}{2}}\iint_{\Sigma'} n'\zeta\, d\Sigma$ also vanish in the limit.

We have now to consider the contribution of the type III solutions to $r^{\frac{1}{2}}\iint_{\Sigma} m'\eta\, d\Sigma$. η involves the functions $e^{kx}G_n{}^m$, and it follows from (11.21) that it is only necessary to examine the solutions which lead to terms in η of the form $e^{kx}G_n$. These are the solutions obtained by putting $m = 1$ in equations (4.83), and in this case we find

$$\eta = \frac{n(n+1)(n+2)\gamma_n{}^1 k e^{kx}}{2(2n+1)}[-G_{n-1} + G_{n+1}]$$
$$+ \frac{n(n+1)(n+2)\gamma_n{}^1 k e^{kx}}{2(2n+3)}[-G_n + G_{n+2}]$$
$$+ \text{ terms of the type } e^{kx}G_n{}^2.$$

It follows from (11.27) that $r^{\frac{1}{2}}\iint_{\Sigma'} m'\eta\, d\Sigma$ vanishes in the limit.

In the same way we can show that the contribution of the type I solutions to $r^{\frac{1}{2}} \iint_{\Sigma'} n'\zeta \, d\Sigma$ also vanishes.

Consider next the contribution of the type IV solutions to $r^{\frac{1}{2}} \iint_{\Sigma'} m'\eta \, d\Sigma$. Here again η involves the functions $e^{kx} G_n^{\ m}$, and it is only necessary to examine the solutions which lead to terms in η of the form $e^{kx} G_n$. These are the solutions obtained by putting $m = 0$ in equations (4.84), and in this case we find

$$\eta = \frac{n\,(n+1)\delta_n k e^{kx}}{2\,(2n+1)}\,[-G_{n-1} + G_{n+1}] + \frac{(n+1)\,(n+2)\,\delta_n k e^{kx}}{2(2n+3)}\,[-G_n + G_{n+2}]$$

$$+ \frac{(n+1)\,\delta_n k e^{kx}}{2n+1}\,[nG_{n-1} - (2n+1)\,G_n + (n+1)\,G_{n+1}]$$

$$- \frac{(n+1)\,\delta_n k e^{kx}}{2n+3}\,[(n+1)\,G_n - (2n+3)\,G_{n+1} + (n+2)\,G_{n+2}]$$

$$+ \text{ terms of the type } e^{kx} G_n^{\ 2}.$$

It follows from (11.27) and (11.28) that $r^{\frac{1}{2}} \iint_{\Sigma'} m'\eta \, d\Sigma$ vanishes in the limit.

In the same way we can show that the contribution of the type II solutions to $r^{\frac{1}{2}} \iint_{\Sigma'} n'\zeta \, d\Sigma$ also vanishes.

This completes the discussion of the complementary function, which contributes nothing to W.

We now see that, as stated above, the only solution which contributes to W is the one associated with the lift in the direction of the y-axis. It follows that equation (11.40) holds good for any motion of the fluid. Combining this result with (10.6), we have

$$Y = -\,2\pi^{\frac{1}{2}} \lim_{r\to\infty} \rho U I R^{-\frac{1}{2}},$$

where

$$R = U/r\nu,$$

i.e., R is a Reynolds number for the sphere Σ.

Similarly

$$Z = -\,2\pi^{\frac{1}{2}} \lim_{r\to\infty} \rho U I' R^{-\frac{1}{2}}.$$

The above discussion shows that trailing vortices are actually given by the present theoretical treatment in the region at a great distance from the body for which it is valid. Although in this region the vorticity has become widely diffused, enough of its characteristics persist to give opposite circulations round two complementary halves of the wake, which die out as $r^{-\frac{1}{2}}$ as we go away from the body. Further, the lift is definitely connected with these circulations, and it is easily seen that the relation between the signs of the lift and the circulations is that required by observation.

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